Functional Analysis, BSM, Spring 2012

Exercise sheet: Operators on Hilbert spaces

Adjoint operator: for every bounded operator $T : H \to H$ there exists a unique bounded operator called the *(Hilbert) adjoint* of T and denoted by T^* such that $(Tx, y) = (x, T^*y)$ for all $x, y \in H$.
The adjoint has the following properties:

$$(S+T)^* = S^* + T^*; (\alpha T)^* = \overline{\alpha} T^*; (ST)^* = T^* S^*; (T^*)^{-1} = \left(T^{-1}\right)^*; (T^*)^* = T; \|T^*\| = \|T\|; \|T^*T\| = \|T\|^2.$$

Definition: An operator $T \in B(H)$ is called

- *self-adjoint* if $T = T^*$;
- *normal* if $TT^* = T^*T$;
- *unitary* if $TT^* = T^*T = I$, that is, $T^{-1} = T^*$;
- *positive* if $(Tx, x) \geq 0$ for all $x \in H$.

Note that self-adjoint and unitary operators are normal.

1. Let $H = \mathbb{C}^n$ with the usual inner product and $T : \mathbb{C}^n \to \mathbb{C}^n$ an arbitrary operator (linear transformation). We denote the corresponding $n \times n$ matrix by M.
a) What is the matrix corresponding to the adjoint operator T^*?
b) What can you say about M for a self-adjoint operator T?

2. Let T be the following $\ell_2 \to \ell_2$ operator:

$$T(\alpha_1, \alpha_2, \ldots) = \left(\frac{\alpha_1 + \alpha_2}{2}, \frac{\alpha_2 + \alpha_3}{2}, \frac{\alpha_3 + \alpha_4}{2}, \ldots \right).$$

Determine the adjoint operator T^*.

3. Let $T \in B(H)$. Show that TT^* and T^*T are both self-adjoint operators.

4. W10P3. (8 points) Let H be a **complex** Hilbert space, $T \in B(H)$. Prove that the following are equivalent:

- T is self-adjoint;
- (Tx, x) is real for all $x \in X$.

5.* Let $T \in B(H)$ be a self-adjoint operator. Prove that

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)|.$$

6. W10P4. (5 points)
a) Let $T \in B(H)$ be a self-adjoint operator. Use the previous exercise to show that if $(Tx, x) = 0$ for all $x \in H$, then $T = 0$.
b) Prove that an operator $T \in B(H)$ is normal if and only if

$$\|Tx\| = \|T^*x\| \text{ for all } x \in H.$$

7.* Prove the *Hellinger-Toeplitz theorem*: if the linear operators $S, T : H \to H$ satisfy $(Sx, y) = (x, Ty)$ for all $x, y \in H$, then S and T are necessarily bounded (and thus $S^* = T; T^* = S$).

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`