Functional Analysis, BSM, Spring 2012

Exercise sheet: Orthonormal bases

Orthonormal system: a subset S of a Hilbert space H is called an *orthonormal system* if $(s,s)=1$ for all $s\in S$ and $(s,s')=0$ for all $s,s'\in S; s\neq s'$.
Orthonormal basis: we say that S is a *complete orthonormal system* (or an *orthonormal basis*) if it is an orthonormal system and there is no orthonormal system S' for which $S'\supsetneq S$.
(By Zorn's lemma every Hilbert space has an orthonormal basis.)
Bessel's inequality: if S is an orthonormal system, then for any $x\in H$

$$\sum_{s\in S}|(x,s)|^2\leq\|x\|^2.$$

Theorem: if S is an orthonormal **basis**, then for any $x\in H$

$$x=\sum_{s\in S}(x,s)s \qquad\text{and}\qquad \|x\|^2=\sum_{s\in S}|(x,s)|^2.$$

Corollary: every Hilbert space is isomorphically isometric to one of the spaces below: for a set Γ let

$$\ell_2(\Gamma)\stackrel{\text{def}}{=}\left\{c:\Gamma\to\mathbb{C}: c_\gamma:=c(\gamma)=0\text{ for all but countably many }\gamma\in\Gamma\text{ and }\sum_{\gamma\in\Gamma}|c_\gamma|^2<\infty\right\}$$

and for $c,c'\in\ell_2(\Gamma)$ let

$$(c,c')\stackrel{\text{def}}{=}\sum_{\gamma\in\Gamma}c_\gamma\overline{c'_\gamma}.$$

(For example, $\ell_2(\mathbb{N})=\ell_2$.) Let H be a Hilbert space and $S=\{s_\gamma:\gamma\in\Gamma\}$ an orthonormal basis of S. Then $H\cong\ell_2(\Gamma)$.

1. Let $S=\{s_1,s_2,\dots\}$ be an orthonormal system of a Hilbert space H.
a) Prove that if $\sum_{i=1}^\infty|\alpha_i|^2<\infty$ for some $\alpha_i\in\mathbb{C}$, then the sum

$$\sum_{i=1}^\infty\alpha_i s_i$$

is convergent in H.
b) Suppose that $\sum_{i=1}^\infty|\alpha_i|^2<\infty$ and $\sum_{i=1}^\infty|\beta_i|^2<\infty$ for some $\alpha_i,\beta_i\in\mathbb{C}$. Then by part a) there exist $x,y\in H$ such that

$$x=\sum_{i=1}^\infty\alpha_i s_i\text{ and }y=\sum_{i=1}^\infty\beta_i s_i.$$

Show that

$$(x,y)=\sum_{i=1}^\infty\alpha_i\overline{\beta_i}.\text{ In particular, }\|x\|^2=\sum_{i=1}^\infty|\alpha_i|^2.$$

2. W10P1. (7 points) Let H be an infinite dimensional Hilbert space. Prove that the following are equivalent.

(i) H is separable.

(ii) Every orthonormal basis of H is countable.

(iii) There exists a countable orthonormal basis of H.

3. Let $S \subset H$ be an orthonormal system. Then the following are equivalent.

(i) S is a complete orthonormal system.

(ii) $x \perp s$ $(\forall s \in S) \implies x = 0$.

(iii) $\mathrm{cl}(\mathrm{span}\, S) = H$.

(iv) $x = \sum_{s \in S} (x, s) s$ for all $x \in H$.

(v) $(x, y) = \sum_{s \in S} (x, s)(s, y)$ for all $x, y \in H$.

(vi) $\|x\|^2 = \sum_{s \in S} |(x, s)|^2$ for all $x \in H$.

Prove $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$.

4.* Let S and T be orthonormal bases of a Hilbert space H. Prove that they must have the same cardinality. (The cardinality of an orthonormal basis is called the dimension of the Hilbert space.)

5. The goal of this exercise is to show that $L_2[0, 1] \cong \ell_2$ by finding a countable orthonormal basis of $L_2[0, 1]$. We define the so-called *Haar functions*: for a nonnegative integer n and an integer $0 \leq k \leq 2^n - 1$ let

$$\Psi_{n,k}(x) = \begin{cases} 2^{n/2}, & \text{if } k2^{-n} \leq x < (k + 1/2)2^{-n} \\ -2^{n/2}, & \text{if } (k + 1/2)2^{-n} \leq x < (k + 1)2^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Let S denote the set of all Haar functions and the constant 1 function on $[0, 1]$.
a) Show that S is an orthonormal system in $L_2[0, 1]$.
b) Suppose that for $f \in L_2[0, 1]$ we have $f \perp g$ for all $g \in S$. Prove that $\int_0^x f \, d\lambda = 0$ for all $x \in [0, 1]$. (Hint: first prove it for $x = k2^{-n}$, then use Lebesgue's dominated convergence theorem.)
c) Show that S is an orthonormal basis in $L_2[0, 1]$.

6.* W10P2. (12 points) Let H be an infinite dimensional Hilbert space. Show that there exists a simple continuous curve $\gamma : [0, 1] \to H$ with the property that any two non-overlapping chords of γ are orthogonal, that is, for any $0 \leq a < b \leq c < d \leq 1$ we have $\gamma(b) - \gamma(a) \perp \gamma(d) - \gamma(c)$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`