Functional Analysis, BSM, Spring 2012

Exercise sheet: Hilbert spaces

Solutions

1. If $x \perp y$, then $(x,y)=0$ and $(y,x)=\overline{(x,y)}=\overline{0}=0$, so

$$\|x+y\|^2=(x+y,x+y)=(x,x)+(x,y)+(y,x)+(y,y)=\|x\|^2+\|y\|^2.$$

2. Let $X=\ell_\infty$,

$$Y=\{(\alpha,0,0,\ldots):\alpha\in\mathbb{C}\}\le X$$

and $x_0:=(0,1,0,0,\ldots)$. Clearly, $d(x_0,Y)=1$ and for any $y=(\alpha,0,0,\ldots)$ with $|\alpha|\le 1$ we have $\|x_0-y\|=1$.
Other example:

$$X=\ell_1; Y=\{(\alpha,\alpha,0,\ldots):\alpha\in\mathbb{C}\}\le X; x_0=(1,-1,0,0,\ldots).$$

3. a) Let $x\in H$ be arbitrary. Since M is a closed linear subspace, by Riesz-lemma there exist $x_1\in M$ and $x_2\in M^\perp$ such that $x=x_1+x_2$. Then

$$x\in\left(M^\perp\right)^\perp \Leftrightarrow (x,y_2)=0 \text{ for all } y_2\in M^\perp \Leftrightarrow (x_1+x_2,y_2)=0 \text{ for all } y_2\in M^\perp$$

Since $x_1\perp y_2$, this is equivalent with

$$(x_2,y_2)=0 \text{ for all } y_2\in M^\perp \Leftrightarrow x_2=0 \Leftrightarrow x\in M.$$

b) We have seen earlier that $M^\perp=(\mathrm{cl}(\mathrm{span}\,M))^\perp$. It follows that

$$\left(M^\perp\right)^\perp=\left((\mathrm{cl}(\mathrm{span}\,M))^\perp\right)^\perp=\mathrm{cl}(\mathrm{span}\,M),$$

where in the last step we used part a) for the closed linear subspace $\mathrm{cl}(\mathrm{span}\,M)$.

4. First suppose that Y is a closed subspace. Then Y is a Hilbert space itself, so the Riesz representation theorem tells us that $\Lambda=\Lambda_y$ for some $y\in Y$. Also, $\widetilde{\Lambda}=\Lambda_z$ for some $z\in X$. We know that

$$\|y\|=\|\Lambda_y\|=\|\Lambda\|=\|\widetilde{\Lambda}\|=\|\Lambda_z\|=\|z\|, \text{ thus } \|y\|=\|z\|.$$

On the other hand, $\widetilde{\Lambda}$ is an extension of Λ, so they coincide on Y, which yields that

$$(u,y)=(u,z) \text{ for all } u\in Y.$$

This means that $(u,z-y)=0$ for all $u\in Y$, so $z-y\in Y^\perp$. In particular, $z-y\perp y$. Then we get $\|z\|^2=\|y\|^2+\|z-y\|^2$ by Pythagorean theorem. However, $\|y\|=\|z\|$, so $\|z-y\|^2=0$, thus z must be equal to y, the extension is indeed unique.

If Y is not closed, then we first extend Λ to a bounded linear functional on $\mathrm{cl}\,Y$. (Such an extension is unique, because Y is dense in $\mathrm{cl}\,Y$ and bounded linear functionals are continuous.) Then we can use the above argument for $\mathrm{cl}\,Y$.

5. Let

$$y=\sum_{i=1}^n (x,e_i)e_i\in M.$$

Then for any $1\le j\le n$:

$$(y,e_j)=\left(\sum_{i=1}^n (x,e_i)e_i,e_j\right)=\sum_{i=1}^n (x,e_i)\cdot(e_i,e_j)=(x,e_j).$$

It means that

$$(x-y,e_j)=(x,e_j)-(y,e_j)=0.$$

So $x-y$ is perpendicular to each e_j. Hence $x-y\in M^\perp$. (In other words, y is the orthogonal projection of x onto M.) Pythagorean theorem yields that y is the closest point of M to x: for any $m\in M$ we have $\|x-y+m\|^2=\|x-y\|^2+\|m\|^2\ge\|x-y\|^2$.