Functional Analysis, BSM, Spring 2012

Exercise sheet: Inner product spaces

Solutions

1.

$$|(x_n,y_n)-(x,y)| = |(x_n,y_n)-(x_n,y)+(x_n,y)-(x,y)| \le |(x_n,y_n)-(x_n,y)|+|(x_n,y)-(x,y)| =$$
$$|(x_n,y_n-y)|+|(x_n-x,y)| \le \|x_n\|\|y_n-y\|+\|x_n-x\|\|y\| \to 0.$$

(The last inequality follows from the Cauchy inequality.)

2. a) The parallelogram law is the sum of the following two equations:

$$\|x+y\|^2 = (x+y,x+y) = (x,x)+(x,y)+(y,x)+(y,y);$$
$$\|x-y\|^2 = (x-y,x-y) = (x,x)-(x,y)-(y,x)+(y,y).$$

b) In real spaces we have $(x,y)=(y,x)$, so the difference of the above equations gives the polarisation formula.
In a complex space $(x,y)+(y,x)=(x,y)+\overline{(x,y)}=2\Re(x,y)$. Thus we get

$$\Re(x,y) = \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right).$$

Now we use this formula for x and iy:

$$\Re(x,iy) = \frac{1}{4}\left(\|x+iy\|^2-\|x-iy\|^2\right).$$

However,

$$\Re(x,iy) = \Re\left(-i(x,y)\right) = \Im(x,y), \text{ so}$$
$$(x,y) = \Re(x,y)+i\Im(x,y) = \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\right).$$

3. We have seen that the ℓ_2-norm is induced by the inner product

$$(x,y) = \sum_{n=1}^{\infty}\alpha_n\overline{\beta_n}, \text{ where } x=(\alpha_1,\alpha_2,\ldots) \text{ and } y=(\beta_1,\beta_2,\ldots).$$

We show that all other ℓ_p-norms fail to satisfy the parallelogram law and thus cannot be induced by an inner product. Let $x=(1,0,0,\ldots)$ and $y=(0,1,0,\ldots)$. It is easy to see that for $1\le p<\infty$ we have

$$\|x\|_p = \|y\|_p = 1 \text{ and } \|x+y\|_p = \|x-y\|_p = \sqrt[p]{2},$$

while for $p=\infty$

$$\|x\|_\infty = \|y\|_\infty = \|x+y\|_\infty = \|x-y\|_\infty = 1.$$

Consequently, the parallelogram law only holds for $p=2$.

4. First suppose that $x_n\to x$, that is, $\|x_n-x\|\to 0$. On the one hand,

$$|\|x_n\|-\|x\|| \le \|x_n-x\| \to 0.$$

On the other hand, the Cauchy inequality implies

$$|(x_n-x,y)| \le \|x_n-x\|\|y\| \to 0$$

for any fixed $y\in H$, thus $x_n \xrightarrow{w} x$.

Now we assume that $\|x_n\|\to\|x\|$ and $x_n \xrightarrow{w} x$. Then

$$\|x_n-x\|^2 = (x_n-x,x_n-x) = (x_n,x_n)-(x_n,x)-(x,x_n)+(x,x) = \|x_n\|^2-(x_n,x)-\overline{(x_n,x)}+\|x\|^2,$$

which converges to

$$\|x\|^2-(x,x)-\overline{(x,x)}+\|x\|^2 = \|x\|^2-\|x\|^2-\|x\|^2+\|x\|^2 = 0.$$

5.* We prove the statement for real normed spaces. The complex case is similar. We define the inner product as the polarisation formula suggests:

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

We need to prove that it is indeed an inner product. (It is clear that it induces the original norm.) For any x, y:

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) = (y, x).$$

For any $x \neq 0$:

$$(x, x) = \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) = \frac{1}{4}\|2x\|^2 > 0.$$

For any x_1, x_2, y:

$$(x_1, y) + (x_2, y) = \frac{1}{4}\left(\|x_1+y\|^2 - \|x_1-y\|^2\right) + \frac{1}{4}\left(\|x_2+y\|^2 - \|x_2-y\|^2\right) =$$

$$\frac{1}{4}\left(\left\|\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2} + y\right\|^2 - \left\|\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2} - y\right\|^2 + \left\|\frac{x_1+x_2}{2} - \frac{x_1-x_2}{2} + y\right\|^2 - \left\|\frac{x_1+x_2}{2} - \frac{x_1-x_2}{2} - y\right\|^2\right) =$$

$$\frac{1}{4}\left(2\left\|\frac{x_1+x_2}{2} + y\right\|^2 + 2\left\|\frac{x_1-x_2}{2}\right\|^2 - 2\left\|\frac{x_1+x_2}{2} - y\right\|^2 - 2\left\|\frac{x_1-x_2}{2}\right\|^2\right) =$$

$$\frac{1}{4}\left(2\left\|\frac{x_1+x_2}{2} + y\right\|^2 - 2\left\|\frac{x_1+x_2}{2} - y\right\|^2\right) = 2\left(\frac{x_1+x_2}{2}, y\right).$$

So we obtained that

$$(x_1, y) + (x_2, y) = 2\left(\frac{x_1+x_2}{2}, y\right). \tag{1}$$

Plugging $x_1 = x$ and $x_2 = 0$ we get

$$(x, y) = 2\left(\frac{x}{2}, y\right). \tag{2}$$

First using (2) with $x = x_1 + x_2$ and then (1):

$$(x_1 + x_2, y) = 2\left(\frac{x_1+x_2}{2}, y\right) = (x_1, y) + (x_2, y).$$

The only property of an inner product that remains to verify is that $(\alpha x, y) = \alpha(x, y)$. If α is a positive integer, then one can prove this by induction using additivity:

$$((n+1)x, y) = (nx + x, y) = (nx, y) + (x, y) = n(x, y) + (x, y) = (n+1)(x, y).$$

It follows that it is also true when α is a negative integer, because

$$0 = (0, y) = (nx + (-n)x, y) = (nx, y) + ((-n)x, y) = n(x, y) + ((-n)x, y).$$

Then for $\alpha = 1/n$:

$$(x, y) = \left(n\left(\frac{1}{n}x\right), y\right) = n\left(\frac{1}{n}x, y\right), \text{ so } \left(\frac{1}{n}x, y\right) = \frac{1}{n}(x, y).$$

For arbitrary rational number $\alpha = m/n$:

$$\left(\frac{m}{n}x, y\right) = \left(m\frac{1}{n}x, y\right) = m\left(\frac{1}{n}x, y\right) = \frac{m}{n}(x, y).$$

Finally, for an arbitrary real number α let us pick a sequence of rational numbers α_n converging to α. Obviously, $\alpha_n(x, y) \to \alpha(x, y)$. On the other hand, $\alpha_n(x, y) = (\alpha_n x, y) \to (\alpha x, y)$. (This follows from the continuity of our inner product, which in turn follows from its definition and the fact that the norm is continuous.) The limits must be equal, so $(\alpha x, y) = \alpha(x, y)$. We are done.

6. We know that the induced norm $\|\cdot\|$ on H satisfies the parallelogram law. We claim that the norm $\|\cdot\|_\sim$ on the completion $\widetilde{H}$ also satisfies the parallelogram law. (Then by the previous exercise it follows that $\|\cdot\|_\sim$ is induced by an inner product, thus $\widetilde{H}$ is a Hilbert space.) Let $x, y \in \widetilde{H}$ be arbitary vectors in the completion. Since H is dense in $\widetilde{H}$, there exist vectors $x_n, y_n \in H$ such that $x_n \to x$ and $y_n \to y$. We know that

$$\|x_n + y_n\|^2 + \|x_n - y_n\|^2 = 2\|x_n\|^2 + 2\|y_n\|^2,$$

which clearly converges to

$$\|x + y\|_\sim^2 + \|x - y\|_\sim^2 = 2\|x\|_\sim^2 + 2\|y\|_\sim^2.$$

7. a) If $x, y \in M^\perp$, then for any $m \in M$

$$(x + y, m) = (x, m) + (y, m) = 0 + 0 = 0,$$

thus $x + y \in M^\perp$. If α is an arbitrary scalar, then

$$(\alpha x, m) = \alpha(x, m) = \alpha \cdot 0 = 0$$

for all $m \in M$, so $\alpha x \in M^\perp$, too. Hence $M^\perp$ is a linear subspace. To see that it is closed, let us suppose that $x_n \in M^\perp$ for all n and $x_n \to x$. Then $(x_n, m) = 0$ for all $m \in M$ and $n \in \mathbb{N}$. Since $(x_n, m) \to (x, m)$ by Exercise 1, it follows that $(x, m) = 0$ for all $m \in M$, thus $x \in M^\perp$.
b) It is clear from the definition that if $A \subset B$, then $A^\perp \supset B^\perp$. Consequently,

$$M^\perp \supset (\mathrm{cl}(\mathrm{span}\, M))^\perp.$$

Now let $x \in M^\perp$ be arbitrary. Then $\{x\}^\perp \supset M$. However, $\{x\}^\perp$ is a closed linear subspace by part a). It follows that $\{x\}^\perp \supset \mathrm{cl}(\mathrm{span}\, M)$, which means that

$$x \in (\mathrm{cl}(\mathrm{span}\, M))^\perp.$$

This proves that

$$M^\perp \subset (\mathrm{cl}(\mathrm{span}\, M))^\perp.$$

c) Let $m \in M$ be arbitrary. By definition, m is perpendicular to each vector $x \in M^\perp$. Consequently, $m \in \left(M^\perp\right)^\perp$ and we are done.