Functional Analysis, BSM, Spring 2012

Exercise sheet: Compact operators

Definition: Let X, Y be Banach spaces. A linear operator $T : X \to Y$ is said to be *compact* if it satisfies the following equivalent conditions: 1. $T(B_1(0))$ is totally bounded; 2. $\mathrm{cl}(T(B_1(0)))$ is compact; 3. for any bounded sequence (x_n) in X the sequence (Tx_n) has a convergent subsequence.
Theorem: Let X be a Banach space and $T : X \to X$ a compact operator.
If $\lambda \neq 0$, then $\lambda I - T$ is surjective if and only if it is injective.
The spectrum of T is finite or countably infinite. Every nonzero element of the spectrum is an eigenvalue with finite multiplicity. The only possible limit point of the spectrum is 0.

1. Let (X, d) be a metric space. Prove that if there exist $c > 0$ and $x_1, x_2, \ldots \in X$ such that $d(x_n, x_m) \geq c$ for any $n \neq m$, then X is not totally bounded. Prove that the converse is also true.

2. a) Show that the unit ball of ℓ_∞ is not totally bounded.
b) Show that the unit ball of ℓ_1 is not totally bounded.
c)* Show that the unit ball of any infinite dimensional normed space is not totally bounded.

3. W7P1. (5 points) Consider the following $T : \ell_\infty \to \ell_\infty$ operator:

$$T : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) \mapsto \left(\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, \frac{\alpha_4}{4}, \ldots\right).$$

Show that T is compact. Determine $\sigma_p(T)$, $\sigma(T)$ and $\sigma_r(T)$.

4. The operator in the previous exercise can be considered also as an $\ell_2 \to \ell_2$ and as an $\ell_1 \to \ell_1$ operator. Are these operators compact, too?

5. Let X, Y be Banach spaces.
a) Let $T_n, T \in B(X, Y)$. Suppose that T_n is compact for each n and $\|T_n - T\| \to 0$ as $n \to \infty$. Prove that T is also compact.
b) An operator $T : X \to Y$ is said to have finite rank if $\mathrm{ran}\, T$ is finite dimensional. Show that every bounded operator with finite rank is compact.
c) Give another proof for the compactness of the operators in Exercise 3, 4.

6. W7P2. (10 points) Consider the space $C[0,1]$ with the supremum norm and let T denote the Volterra integral operator:

$$(Tf)(x) \overset{\text{def}}{=} \int_0^x f(y)\,\mathrm{d}y.$$

Prove that T is a compact operator.

7.* Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a fixed continuous function. Consider the space $C[0,1]$ with the supremum norm. The integral operator with kernel k is defined as

$$(Tf)(x) \overset{\text{def}}{=} \int_0^1 k(x,y) f(y)\,\mathrm{d}y.$$

Prove that T is compact for any continuous kernel k.

8. W7P3. (8 points)
a) Let X, Y, Z be Banach spaces; $T \in B(X, Y)$ and $S \in B(Y, Z)$. Prove that the operator $ST : X \to Z$ is compact if either S or T is compact.
b) The converse is not true: find a Banach space X and a bounded operator $T : X \to X$ such that T^2 is compact, but T is not.

9. Let X be an infinite dimensional Banach space and $T : X \to X$ a compact operator. Show that T is not surjective. It follows that $0 \in \sigma(T)$. Find examples when $0 \in \sigma_p(T)$, when $0 \in \sigma_r(T)$ and when $0 \in \sigma_c(T) = \sigma(T) \setminus \sigma_p(T) \setminus \sigma_r(T)$.

10. Show that if a compact operator T has infinite rank, then T is not bounded below.

11. Let $\tau : [0,1] \to [0,1]$ be a fixed continuous function. Consider the following operator on $C[0,1]$: $Tf \overset{\text{def}}{=} f \circ \tau$. For which τ is T compact?

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`