# Functional Analysis, BSM, Spring 2012

Exercise sheet: Spectra of operators

Solutions

**1.** We proved earlier that $\ker T$ is a linear subspace. Since $T$ is bounded, it is continuous, so the preimage of any closed set is closed. However, $\ker T$ is the preimage of $\{0\} \subset Y$, which is clearly a closed set.

**2.** a) If $y_1, y_2 \in \operatorname{ran} T$, then $\exists x_1, x_2 \in X$ with $Tx_1 = y_1$ and $Tx_2 = y_2$. Thus $y_1 + y_2 = T(x_1 + x_2) \in \operatorname{ran} T$. If $\alpha \in \mathbb{C}$, then $\alpha y_1 = T(\alpha x_1) \in \operatorname{ran} T$.
b) Let $y = (1, 1/2, 1/3, \ldots)$ and let $y_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$. It is easy to check that $y, y_n \in \ell_2$ and $\|y - y_n\|_2 \to 0$ as $n \to \infty$. However, $y_n \in \operatorname{ran} T$, but $y \notin \operatorname{ran} T$, which implies that $\operatorname{ran} T$ is not closed.

**3.** Let $y_1, y_2, \ldots \in \operatorname{ran} T$ converging to $y \in Y$. We need to show that $y \in \operatorname{ran} T$, too. There exists $x_n \in X$ such that $Tx_n = y_n$. Since $T$ is bounded below, we have

$$\|x_n - x_m\| \leq \frac{1}{c}\|Tx_n - Tx_m\| = \frac{1}{c}\|y_n - y_m\|.$$

However, $(y_n)$ is Cauchy (because it is convergent), thus so is $(x_n)$. Since $X$ is complete, $(x_n)$ is convergent: $\|x_n - x\| \to 0$ as $n \to \infty$. Using that $T$ is continuous, we get that $Tx_n = y_n$ converges to $Tx$. Thus $y = Tx$; it follows that $y \in \operatorname{ran} T$.

**4.** First suppose that $T$ is invertible. Then $T$ is surjective, so $\operatorname{ran} T = Y$ is indeed dense. Since $T^{-1}$ is bounded, we get

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\|,$$

which implies that $\|Tx\| \geq c\|x\|$ with $c = 1/\|T^{-1}\|$.

Now suppose that $T$ is bounded below and $\operatorname{ran} T$ is dense. By the previous exercise $\operatorname{ran} T$ must be closed, thus $\operatorname{ran} T = X$, that is, $T$ is surjective. Also, $T$ is injective, because if $Tx = 0$, then $\|x\| \leq \|Tx\|/c = 0$, so $x = 0$. Consequently, $T$ is bijective. By the inverse mapping theorem it follows that $T$ is invertible.

**5.** a) We saw earlier that $\|T\| = 1$ and that $\sigma_p(T)$ (the set of eigenvalues) is the closed unit disk $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. (The vector $(1, \lambda, \lambda^2, \ldots) \in \ell_\infty$ is an eigenvector for $\lambda$.)
b) It holds for arbitrary $T$ that

$$\sigma_p(T) \subset \sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}.$$

Here both the left-hand side and the right-hand side are the closed unit disk. It follows that $\sigma(T)$ is also the closed unit disk. Finally, the residual spectrum is empty, because $\sigma_r(T) \subset \sigma(T) \setminus \sigma_p(T)$.

**6.** a) Since $\|T\| = 1$, $\sigma(T)$ is contained by the closed unit disk. On the other hand, $\sigma_p(T)$ is the open unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$; $\sigma_p(T) \subset \sigma(T)$ and $\sigma(T)$ is closed, so $\sigma(T)$ must contain the closure of $\sigma_p(T)$, which is the closed unit disk again. Hence $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
b) For the operator $I - T$ we have

$$I - T : x = (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_4, \ldots).$$

For given $\beta_, \ldots, \beta_n$, we need to solve the equation $(I-T)x = (\beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots)$. We get that $\alpha_2 = \alpha_1 - \beta_1$, $\alpha_3 = \alpha_1 - \beta_1 - \beta_2$, and so on. For $m > n$ we get

$$\alpha_m = \alpha_1 - \beta_1 - \beta_2 - \cdots - \beta_n.$$

So if we set $\alpha_1 = \beta_1 + \cdots + \beta_n$, then $\alpha_m = 0$ for all $m > n$ and we get a solution $x \in \ell_1$.
c) Since $1 \in \sigma(T)$, $I - T$ is not bijective. Since $1 \notin \sigma_p(T)$, $I - T$ is injective. Consequently, $I - T$ cannot be surjective: $\operatorname{ran}(I - T) \neq \ell_1$. Actually, it is not hard to show that

$$y = \left(\frac{1}{1 \cdot 2}, \frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \ldots\right) \notin \operatorname{ran}(I - T).$$

We will use that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

When we solve $(I-T)x=y$, then we get that

$$\alpha_{n+1}=\alpha_1-1+\frac{1}{n+1}.$$

We would need a solution for which $\sum_n |\alpha_n|<\infty$. We can only hope this if $\alpha_n\to 0$ as $n\to\infty$. Consequently, we have to set $\alpha_1=1$. Then $\alpha_{n+1}=1/(n+1)$. However, the sum of these is infinite. Thus we proved that there is no $x\in\ell_1$ with $(I-T)x=y$; so $y\notin \operatorname{ran} T$.
d) Since $\sigma_r(T)\subset\sigma(T)\setminus\sigma_p(T)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$, we need to check the complex numbers of unit length. We proved that $\operatorname{ran}(I-T)$ is dense. Basically the same proof shows that $\operatorname{ran}(\lambda I-T)$ is dense for any $|\lambda|=1$. It follows that $\sigma_r(T)=\emptyset$.

**7.** If such a $\Lambda$ exists, then $\operatorname{ran} T\subset\ker\Lambda$. However, $\ker\Lambda$ is a closed proper subspace of $X$, so the closure of $\operatorname{ran} T$ is also contained by $\ker\Lambda$, so it cannot be the whole space, $\operatorname{ran} T$ is not dense.

To prove the other direction, suppose that $\operatorname{ran} T$ is not dense. Then the closure of $\operatorname{ran} T$ is a closed proper subspace $Y\leq X$. Pick some $x\in X\setminus Y$. Using the Hahn-Banach theorem it is not hard to prove the existence of a bounded linear functional $\Lambda\in X^*$ for which $\Lambda y=0$ for $y\in Y$ and $\Lambda x=1$. Then $\Lambda\neq 0$, but $\Lambda T=0$.

**8.** a) We saw earlier that $\|T\|=1$ and $\sigma_p(T)=\emptyset$.
b) For $|\lambda|\leq 1$ consider the vector

$$y=(1,\lambda,\lambda^2,\lambda^3,\ldots)\in\ell_\infty$$

and the corresponding bounded linear functional $\Lambda_y\in\ell_1^*$. We claim that $\Lambda_y(\lambda I-T)=0$. Indeed, for an arbitrary $x=(\alpha_1,\alpha_2,\ldots)\in\ell_1$:

$$\Lambda_y(\lambda I-T)x=\Lambda_y(\lambda\alpha_1,\lambda\alpha_2-\alpha_1,\lambda\alpha_3-\alpha_2,\ldots)=\lambda\alpha_1+\lambda(\lambda\alpha_2-\alpha_1)+\lambda^2(\lambda\alpha_3-\alpha_2)+\cdots=0.$$

By the previous exercise it follows that $\operatorname{ran}(\lambda I-T)$ is not dense, so $\lambda\in\sigma_r(T)$ for $|\lambda|\leq 1$.
c) $\sigma_r(T)=\sigma(T)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$.

**9.** a) Let us consider the ball $B$ with radius $1/2$ and center $(1,1,\ldots)\in\ell_\infty$ It consists of points $y=(\beta_1,\beta_2,\ldots)$ with $|\beta_n-1|<1/2$ for all $n$. We will only use that the real part $\Re\beta_n$ is at least $1/2$ for all $n$. We claim that for any such $y$ there is no $x\in\ell_\infty$ such that $(I-T)x=y$, that is $\operatorname{ran}(I-T)$ is disjoint from $B$. Assume that $(I-T)x=y$ for some $x=(\alpha_1,\alpha_2,\ldots)$. We get that $\alpha_1=\beta_1$, $\alpha_2=\beta_1+\beta_2$, $\alpha_3=\beta_1+\beta_2+\beta_3$, and so on. It follows that $\Re\alpha_n\geq\Re\beta_1+\cdots+\Re\beta_n\geq n/2$, which contradicts that $x=(\alpha_1,\alpha_2,\ldots)\in\ell_\infty$.
b) $\|T\|=1$; $\sigma_p(T)=\emptyset$. We claim that $\sigma_r(T)=\sigma(T)$ is the closed unit ball. For $|\lambda|<1$, the same argument works as in the previous exercise: $y=(1,\lambda,\lambda^2,\ldots)\in\ell_1$, so $\Lambda_y\in\ell_\infty^*$. It is easy to check that $\Lambda_y(\lambda I-T)=0$, so $\operatorname{ran}(\lambda I-T)$ is not dense; $\lambda\in\sigma_r(T)$. If $|\lambda|=1$, then one can easily generalize the argument in a) to find a ball that is disjoint from $\operatorname{ran}(\lambda I-T)$. Again, it follows that $\operatorname{ran}(\lambda I-T)$ is not dense, $\lambda\in\sigma_r(T)$.

**10.** a) Suppose that $(I-T)x=y$ for some $x=(\alpha_1,\alpha_2,\ldots)\in\ell_2$ and $y=(\beta_1,\beta_2,\ldots)\in\ell_2$. It can be seen easily that

$$\alpha_n=\beta_1+\beta_2+\cdots+\beta_n.$$

So for $y=(1,1/2,1/4,\ldots)\in\ell_2$ there exists no such $y\in\ell_2$. This shows that $\operatorname{ran}(I-T)\neq\ell_2$. Now we prove that $\operatorname{ran}(I-T)$ is dense. Clearly $\operatorname{ran}(I-T)$ contains those vectors $y=(\beta_1,\ldots,\beta_N,0,0,\ldots)$ for which $\beta_1+\cdots+\beta_N=0$. So it suffices to show that the set of such vectors is dense in $\ell_2$. The key idea here is that the sum of positive reals can be arbitrarily large while their square sum is arbitrarily small:

$$\sum_{n=N+1}^{\infty}\frac{1}{n}=\infty,\text{ but }\sum_{n=N+1}^{\infty}\frac{1}{n^2}\to 0\text{ as }N\to\infty.$$

So if some $x=(\alpha_1,\alpha_2,\ldots)\in\ell_2$ and $\varepsilon>0$ are given, then first we pick $m$ such that $\|x-x_m\|<\varepsilon/2$ for $x_m=(\alpha_1,\ldots,\alpha_m,0,0,\ldots)$. Then we replace finitely many of the 0's by $\gamma_1,\ldots,\gamma_k$ such that

$$\sqrt{\sum_{i=1}^{k}|\gamma_i|^2}<\frac{\varepsilon}{2}\text{ and }\alpha_1+\cdots+\alpha_m+\gamma_1+\cdots\gamma_k=0.$$

Then $x'_m = (\alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_k, 0, 0, \ldots)$ has the desired form and $\|x - x'_m\| \leq \|x - x_m\| + \|x_m - x'_m\| < \varepsilon$.
b) It is easy that $\|T\| = 1$, $\sigma_p(T) = \emptyset$. If $|\lambda| < 1$, then $\Lambda_y(\lambda I - T) = 0$, where $y = (1, \lambda, \lambda^2, \ldots) \in \ell_2$. It follows that $\mathrm{ran}(\lambda I - T)$ is not dense, so $\lambda \in \sigma_r(T)$. If $|\lambda| = 1$, then $\mathrm{ran}(\lambda I - T)$ is dense (the proof is basically the same as for $\lambda = 1$). It means that $\lambda \notin \sigma_r(T)$. Consequently, $\sigma_r(T)$ is the open unit disc, while $\sigma(T)$ is the closed unit disc.

**11.** Pick an arbitrary $f \in C[0,1]$ with $\|f\| \leq 1$. Then $f(x) \leq 1$ for all $x \in [0,1]$. It follows that $(Tf)(x) \leq x$, $(T^2 f)(x) \leq x^2/2$, $(T^3 f)(x) \leq x^3/6$, and so on. One can show by induction that $(T^k f)(x) \leq x^k/k!$. Similarly, since $f(x) \geq -1$ for all $x$, we obtain that $(T^k f)(x) \geq -x^k/k!$. It follows that $\|T^k f\| \leq 1/k!$ for any $f$ with $\|f\| \leq 1$. It means that the operator norm of $T^k$ is at most $1/k!$. In fact, the constant 1 function shows that $\|T^k\| = 1/k!$. Thus $\|T\| = 1$ and

$$r(T) = \inf_k \sqrt[k]{\|T^k\|} = \inf_k \frac{1}{\sqrt[k]{k!}} = 0.$$

The kernel of $T$ is trivial (i.e., $\ker T = \{0\}$), since $Tf = 0$ implies that $f = 0$ (note that $Tf$ is differentiable and its derivative is $f$). So $T$ is injective. It is clearly not surjective, since $Tf$ is always 0 at 0. Thus $0 \in \sigma(T)$. The spectrum has no other point, because it is contained by $\{\lambda : |\lambda| \leq r(T)\} = \{0\}$. So $\sigma(T) = \{0\}$. Finally, we show that the range is not closed. It is not hard to see that $\mathrm{ran}\, T$ is the set of continuously differentiable functions $g$ with $g(0) = 0$. A sequence of such functions can clearly converge (in the supremum norm) to a non-differentiable function.

**12.** We proved earlier that $\|S_1 S_2\| \leq \|S_1\| \cdot \|S_2\|$, where $S_1 S_2$ is the composition of $S_1$ and $S_2$. Since $T^{m+n}$ is the composition of $T^m$ and $T^n$:

$$\|T^{m+n}\| = \|T^m T^n\| \leq \|T^m\| \cdot \|T^n\|.$$

Taking logarithms of both sides: $a_{m+n} \leq a_m + a_n$. It remains to show that for any such sequence

$$\lim_{k\to\infty} a_k/k = \inf_k a_k/k.$$

Clearly, $\liminf_{k\to\infty} a_k/k \geq \inf_k a_k/k$; it suffices to show that $\limsup_{k\to\infty} a_k/k \leq \inf_k a_k/k$. We need that for any fixed $m$ we have $\limsup_{k\to\infty} a_k/k \leq a_m/m$. Any $k$ can be written as $sm + r$ with $0 \leq r < m$. We know that $a_k = a_{sm+r} \leq a_{sm} + a_r \leq s \cdot a_m + a_r$. Thus

$$\frac{a_k}{k} \leq \frac{s \cdot a_m}{k} + \frac{a_r}{k} \leq \frac{s \cdot a_m}{sm} + \frac{a_r}{k} = \frac{a_m}{m} + \frac{a_r}{k}.$$

The right-hand side tends to $a_m/m$ as $k \to \infty$, we are done.

**13.** We use that

$$T^{-1} - S^{-1} = S^{-1}(S - T)T^{-1}.$$

It follows that

$$\|T^{-1} - S^{-1}\| \leq \|S^{-1}\|\|S - T\|\|T^{-1}\| \leq \|S^{-1}\| \frac{1}{2\,\|T^{-1}\|} \|T^{-1}\| = \frac{1}{2}\|S^{-1}\|,$$

which yields that

$$\|T^{-1}\| \geq \|S^{-1}\| - \|T^{-1} - S^{-1}\| \geq \|S^{-1}\| - \frac{1}{2}\|S^{-1}\| = \frac{1}{2}\|S^{-1}\|.$$

**14.** Let $S$ be the left shift operator on $\ell_1$. We notice that $T = 1 + S + S^2$. Let $p(z) = 1 + z + z^2$. Using the spectral mapping theorem and the fact that the spectrum of $S$ is the closed unit disk:

$$\sigma(T) = \left\{1 + z + z^2 : \|z\| \leq 1\right\}.$$

To determine its intersection with the real axis, we need to determine the set of real numbers $c$ for which the equation

$$1 + z + z^2 = c \Leftrightarrow z^2 + z + (1 - c) = 0$$

has a solution with $|z| \leq 1$. Solving this quadratic equation:

$$z = \frac{-1 \pm \sqrt{1 - 4(1 - c)}}{2} = \frac{-1}{2} \pm \sqrt{c - \frac{3}{4}}.$$

It is easy to check that the exact condition of at least one root being in the closed unit disk is that $0 \leq c \leq 3$. So the intersection in question is $[0, 3]$. (Note that we would get a different set if we took the intersection $\sigma(S) \cap \mathbb{R} = [-1, 1]$ and then took the image of this set under $p$, which is $[3/4, 3]$.)

**15.** It is clearly enough to show that $r(TS) \leq r(ST)$. The key observation is the following:

$$(TS)^k = TSTS\cdots TS = T(STST\cdots ST)S = T(ST)^{k-1}S.$$

Then

$$\|(TS)^k\| \leq \|T\|\|(ST)^{k-1}\|\|S\|.$$

Let $\varepsilon > 0$; then for any large enough $k$ we have

$$\sqrt[k-1]{\|(ST)^{k-1}\|} < r(ST) + \varepsilon.$$

Consequently,

$$\|(TS)^k\| \leq \|T\|\|S\| \left(r(ST) + \varepsilon\right)^{k-1} = \frac{\|T\|\|S\|}{r(ST) + \varepsilon} \left(r(ST) + \varepsilon\right)^k.$$

Taking $k$-th root, then taking the limit as $k \to \infty$ we get that $r(TS) \leq r(ST) + \varepsilon$. Since this holds for any $\varepsilon > 0$, it follows that $r(TS) \leq r(ST)$.

**16.** Pick $q \in \mathbb{R}$ such that $r(T) < q < 1$. We know that $\|T^k\| < q^k$ for large enough $k$. We set

$$S_k = I + T + T^2 + \cdots + T^{k-1}.$$

It is easy to see that $S_k$ is a Cauchy sequence in $B(X)$. Since $X$ is complete, so is $B(X)$, which yields that $S_k$ is convergent. Let $S \in B(X)$ denote the the limit of $S_k$, that is, $\|S - S_k\| \to 0$. We need to show that $S(I - T) = (I - T)S = I$. Since

$$S_k(I - T) = (I + T + \cdots + T^{k-1})(I - T) = I - T^k,$$

we have

$$\|S_k(I - T) - I\| = \|T^k\| \to 0 \text{ as } k \to \infty.$$

Consequently,

$$\|S(I - T) - I\| = \|S(I - T) - S_k(I - T) + S_k(I - T) - I\| \leq$$
$$\|(S - S_k)(I - T)\| + \|S_k(I - T) - I\| \leq \|S - S_k\|\|I - T\| + \|S_k(I - T) - I\| \to 0$$

as $k \to \infty$. It follows that $S(I - T) = I$. Proving that $(I - T)S = I$ is similar.

**17.*** a) We know from previous exercises that if $r(ST) < 1 \Leftrightarrow r(TS) < 1$, then both $I - ST$ and $I - TS$ are invertible. However, this does not help us when $r(ST) \geq 1$.

Suppose that $I - ST$ is invertible, let $U \in B(X)$ be the inverse, that is, $U(I - ST) = (I - ST)U = I$. We need to find an inverse operator $V$ for $I - TS$. To get an idea how to define $V$, we consider the case $r(ST) = r(TS) < 1$. Then $U = I + ST + STST + \cdots$ and $V = I + TS + TSTS + \cdots$. Clearly, $V = I + TUS$. So we will define $V$ with this formula in the general case. Then using $U(I - ST) = I$:

$$V(I - TS) = (I + TUS)(I - TS) = I - TS + TUS - TUSTS =$$
$$I + T(-I + U - UST)S = I + S\left(U(I - ST) - I\right)T = I.$$

Proving that $(I - TS)V = I$ is similar.
b) Using the first part we get that for any $\lambda \neq 0$:

$$\lambda \notin \sigma(ST) \Leftrightarrow \lambda I - ST \text{ invertible } \Leftrightarrow I - \frac{S}{\lambda}T \text{ invertible } \Leftrightarrow I - T\frac{S}{\lambda} \text{ invertible } \Leftrightarrow \lambda I - TS \text{ invertible } \Leftrightarrow \lambda \notin \sigma(ST).$$