# Functional Analysis, BSM, Spring 2012

Exercise sheet: norms and bounded operators

Solutions

**1.**

$$0 \le \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \le \|(x_n - x)\| + \|(y_n - y)\|.$$

Both terms on the right-hand side converge to 0, so $\|(x_n + y_n) - (x + y)\|$ converges to 0 as well.

**2.**

$$|\|x_n\| - \|x\|| \le \|x_n - x\| \to 0.$$

**3.** Let us denote the derivative operator by $T$; so $Tf = f'$. We claim that $T$ is not bounded, that is, there is no upper bound for the ratio $\frac{\|Tf\|}{\|f\|}$. In other words, there exist functions $f_n \in C^\infty[0,1]$ such that the ratio $\frac{\|f_n'\|}{\|f_n\|}$ goes to infinity as $n \to \infty$.

Let $f_n(x) = e^{nx}$; then $f_n'(x) = n \cdot e^{nx}$. We have

$$\|f_n\| = \sup_{x\in[0,1]} |e^{nx}| = e^n.$$

Similarly,

$$\|f_n'\| = \sup_{x\in[0,1]} |n \cdot e^{nx}| = n \cdot e^n.$$

Their ratio is $n$, which tends to infinity as we wanted.

**4.** For $x = (\alpha_1, \alpha_2, \ldots) \ne 0$ we have

$$\frac{\|Tx\|_1}{\|x\|_1} = \frac{|\alpha_2| + |\alpha_3| + \ldots}{|\alpha_1| + |\alpha_2| + |\alpha_3| + \ldots}.$$

The numerator is clearly less than or equal to the denominator, so this ratio is always at most 1. This means that $T$ is bounded and $\|T\| \le 1$. To show that $\|T\| = 1$, let $x$ be any element of $\ell_1$ with $\alpha_1 = 0$, for example, let $x = (0, 1/2, 1/4, 1/8, \ldots)$. Then the numerator and the denominator are equal, so in this case the ratio is 1.

**5.** Let $x = (\alpha_1, \alpha_2, \ldots)$. For $p = \infty$ we have $\|Tx\|_\infty = \|x\|_\infty$ for any $x \in \ell_\infty$. This means that $\|T\|_{\infty,\infty} = 1$.

For $1 \le p < \infty$ and $x \in \ell_p$:

$$\|x\|_p = \sqrt[p]{|\alpha_1|^p + |\alpha_2|^p + |\alpha_3|^p + \ldots}$$

and

$$\|Tx\|_p = \sqrt[p]{2|\alpha_1|^p + |\alpha_2|^p + |\alpha_3|^p + \ldots}$$

Clearly,

$$\|Tx\|_p \le \sqrt[p]{2} \cdot \|x\|_p$$

with equality whenever $\alpha_2 = \alpha_3 = \ldots = 0$. It follows that $\|T\|_{p,p} = \sqrt[p]{2}$.

**6.**

$$|\alpha_n - \alpha_{n+1}| \le |\alpha_n| + |\alpha_{n+1}| \le 2 \cdot \sup_i |\alpha_i|,$$

which implies that $\|Tx\| \le 2\|x\|$ for any $x \in X$. We show an $x \in X$ for which we have equality, thus proving that $\|T\| = 2$. Let $x = (1, -1, 0, 0, 0, \ldots)$; then $T(x) = (2, -1, 0, 0, 0, \ldots)$. So $\|x\| = 1$ and $\|Tx\| = 2$.

The operator $S$, on the other hand, is not bounded. Let

$$x = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots) \in X.$$

Then

$$Sx = (n, n-1, n-2, \ldots, 1, 0, 0, \ldots).$$

So $\|Sx\| = n$, while $\|x\| = 1$; thus the ratio is not bounded; $\|S\| = \infty$.

Now we prove that $\ker T = \{0\}$. Suppose that $Tx = 0$ for some $x = (\alpha_1, \alpha_2, \ldots) \in X$. It means that $\alpha_1 - \alpha_2 = \alpha_2 - \alpha_3 = \ldots = 0$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = \ldots$. However, all but finitely many $\alpha_i$'s are zero, so they all must be zero, that is, $x = 0$.

As we have seen, $\ker T = \{0\}$ implies that $T$ is injective. For surjectivity, notice that $S$ is the inverse of $T$, that is, $ST = TS = \text{id}$. So if $x \in X$, then for $y = Sx$ we have $Ty = T(Sx) = x$.

**7.**

$$T^k(\alpha_1,\alpha_2,\alpha_3,\ldots)=(2^k\alpha_1,2^{k-1}\alpha_1,2^{k-2}\alpha_1,\ldots,\alpha_1,\alpha_2,\alpha_3,\ldots),$$

the $\ell_1$-norm of which is

$$\left(2^{k+1}-1\right)|\alpha_1|+|\alpha_2|+|\alpha_3|+\ldots$$

Thus $\|T^kx\|_1\le(2^{k+1}-1)\|x\|_1$ for any $x\in X$ with equality if $\alpha_2=\alpha_3=\ldots=0$. Consequently, $\|T^k\|=2^{k+1}-1$, which yields that

$$2<\sqrt[k]{\|T^k\|}<2\cdot\sqrt[k]{2}.$$

It follows that the limit (as $k\to\infty$) is 2. Thus $\varrho(T)=2$.

For $x=(1,1/2,1/4,1/8,\ldots)$ we have $Tx=2x$, so $\lambda=2$ is an eigenvalue for $T$.

**8.*** By $F_n$ we denote the Fibonacci numbers: $F_0=0$, $F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$, $F_6=8$, and so on. It can be proved easily by induction on $k$ that for $x=(\alpha_1,\alpha_2,\ldots)$:

$$T^kx=\left(F_{k+1}\alpha_1+F_k\alpha_2,F_k\alpha_1+F_{k-1}\alpha_2,\ldots,F_2\alpha_1+F_1\alpha_2,\alpha_1,\alpha_2,\alpha_3,\ldots\right).$$

It follows that $\|T^kx\|_\infty\le(F_{k+1}+F_k)\|x\|_\infty=F_{k+2}\|x\|_\infty$ with equality if $x=(1,1,0,0,\ldots)$. This means that $\|T^k\|_{\infty,\infty}=F_{k+2}$. Using that

$$F_n=\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n-\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$$

we conclude that $\varrho(T)=(1+\sqrt5)/2$.

Now let $\varphi=(1+\sqrt5)/2$; we show that $\varphi$ is an eigenvalue for $T$. Notice that $\varphi^2=\varphi+1$. Using this it is easy to see that

$$x=\left(\varphi,1,\varphi^{-1},\varphi^{-2},\varphi^{-3},\ldots\right)$$

is an eigenvector with eigenvalue $\varphi$.

**9.** Let $x$ be an eigenvector for $\lambda$:

$$\frac{\|Tx\|}{\|x\|}=\frac{\|\lambda x\|}{\|x\|}=\frac{|\lambda|\cdot\|x\|}{\|x\|}=|\lambda|.$$

Since $\|T\|$ is the supremum of $\|Tx\|/\|x\|$, it follows that $|\lambda|\le\|T\|$.

To prove that $|\lambda|\le\varrho(T)$, we need the following observation: if $x$ is an eigenvector for $T$ with eigenvalue $\lambda$, then $x$ is also en eigenvector for $T^k$, but this time with eigenvalue $\lambda^k$. This is easy to prove by induction:

$$T^kx=T\left(T^{k-1}x\right)=T\left(\lambda^{k-1}x\right)=\lambda^{k-1}T(x)=\lambda^{k-1}\cdot\lambda x=\lambda^kx.$$

Applying the already proven first inequality to the operator $T^k$ we get that $|\lambda^k|\le\|T^k\|$. Taking $k$-th root:

$$|\lambda|\le\sqrt[k]{\|T^k\|}.$$

Taking the limit as $k\to\infty$, $|\lambda|\le\varrho(T)$ follows.

**10.** Let $x_n$ be an absolutely summable sequence in a Banach space, and set

$$r_N\stackrel{\text{def}}{=}\sum_{i=N+1}^{\infty}\|x_i\|.$$

Since $\sum_{i=1}^{\infty}\|x_i\|<\infty$, we know that $r_N\to0$ as $N\to\infty$. This means that for any $\varepsilon>0$ there is an $N$ such that $r_N<\varepsilon$. Then for $n\ge m\ge N$:

$$\|s_n-s_m\|=\|x_{m+1}+x_{m+2}+\ldots+x_n\|\le\|x_{m+1}\|+\|x_{m+2}\|+\ldots+\|x_n\|\le r_N<\varepsilon.$$

Consequently, the sequence $s_1,s_2,\ldots$ is Cauchy. Our space is complete, so it follows that $(s_n)$ is convergent. Then, by definition, $(x_n)$ is summable.

**11.*** Let $x_1,x_2,\ldots$ be an arbitrary Cauchy sequence. We need to prove that it is convergent. For $\varepsilon=1/2^k$ let us choose $N_k$ such that $\|x_n-x_m\|\le1/2^k$ if $n,m\ge N_k$. We can assume that $N_{k+1}>N_k$. Consider the sequence $y_k=x_{N_{k+1}}-x_{N_k}$. We claim that this sequence is absolutely summable. Indeed,

$$\|y_k\|=\|x_{N_{k+1}}-x_{N_k}\|\le\frac{1}{2^k},$$

thus

$$\sum_{k=1}^{\infty}\|y_k\|\le\sum_{k=1}^{\infty}\frac{1}{2^k}=1<\infty.$$

Our assumpion was that every absolutely summable sequence is summable, so $(y_k)$ is summable, that is, $(x_{N_k})$ is convergent. We proved that the Cauchy sequence $x_1,x_2,\ldots$ has a convergent subsequence. It follows that $(x_n)$ is convergent, too.