Functional Analysis, BSM, Spring 2012

Exercise sheet: metric spaces and convergence

Solutions

1. We need to show that

$$-d(x_1, x_2) \le d(x_1, y) - d(x_2, y) \le d(x_1, x_2).$$

Both inequalities follow from the triangle inequality.

2. Let

$$S_n = \{(a_1, a_2, \ldots) \in X : a_{n+1} = a_{n+2} = \ldots = 0\}.$$

This is clearly a finite set (its cardinality is 2^n). Moreover, S_n has the property that for any point $x \in X$ there exists a point $y \in S_n$ such that $d(x, y) \le 1/(n+1)$. Indeed: if $x = (a_1, a_2, \ldots)$, then let

$$y = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in S_n.$$

The first n elements are the same for x and y, so $d(x, y) \le 1/(n+1)$.

As for the separability of X, let us consider the set

$$S = S_1 \cup S_2 \cup S_3 \cup \cdots$$

This is clearly a countable dense subset of X: if $x \in X$, then let $x_n \in S_n \subset S$ such that $d(x, x_n) \le 1/(n+1)$. The sequence $x_1, x_2, \ldots \in S$ clearly converges to x. (In fact, with a similar argument one can easily prove that any totally bounded metric space is separable.)

3. We prove by contradiction. Assume that $x \ne y$. Then $d(x, y) > 0$; set $\varepsilon = d(x, y)/2$. Since $x_n \xrightarrow{d} x$, there exists N_1 such that $d(x_n, x) < \varepsilon$ for $n \ge N_1$. Since $x_n \xrightarrow{d} y$, there exists N_2 such that $d(x_n, y) < \varepsilon$ for $n \ge N_2$. Now let n be any integer greater than $\max(N_1, N_2)$; we have

$$d(x, y) \le d(x_n, x) + d(x_n, y) < \varepsilon + \varepsilon = d(x, y),$$

which is a contradiction.

4. We need to prove that $d(x_n, y) - d(x, y) \to 0$. By Exercise 1 we have

$$|d(x_n, y) - d(x, y)| \le d(x_n, x).$$

Since $x_n \xrightarrow{d} x$, the right-hand side converges to 0, hence so does the left-hand side.

5. Let $\varepsilon > 0$ be an arbitrary real number. Since $x_n \xrightarrow{d} x$, there exists N_1 such that $d(x_n, x) < \varepsilon/2$ for $n \ge N_1$. Since $y_n \xrightarrow{d} y$, there exists N_2 such that $d(y_n, y) < \varepsilon/2$ for $n \ge N_2$. Then for any $n \ge \max(N_1, N_2)$ we have

$$|d(x_n, y_n) - d(x, y)| = |d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x, y)| \le$$
$$|d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \le d(y_n, y) + d(x_n, x) < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

This proves that $d(x_n, y_n) \to d(x, y)$.

We can use this fact to give another proof for Exercise 3. Let us consider the special case when $x_n = y_n$: we have $d(x_n, y_n) = 0$, so the limit $d(x, y)$ must be 0, too.

6. Let $\varepsilon > 0$ be an arbitrary real number. Since $x_{k_n} \to x$, there exists N_1 such that $d(x_{k_n}, x) < \varepsilon/2$ for $n \ge N_1$. Since (x_n) is Cauchy, there exists N_2 such that $d(x_m, x_n) < \varepsilon/2$ for $m, n \ge N_2$. Let $N = \max(N_1, N_2)$ and $N' = k_N$. For any $n \ge N'$:

$$d(x_n, x) \le d\left(x_n, x_{k_N}\right) + d\left(x_{k_N}, x\right) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

7. First we give an uncountable subset of ℓ_∞ such that the distance of any two distinct points in the set is at least 1. We will see that the existence of such a set implies that the space is not separable. Let

$$M = \{(a_1, a_2, \ldots) : a_i \in \{0, 1\}\} \subset \ell_\infty.$$

It is a well-known fact that M is an uncountable set. Note that the distance of any two distinct points in M is exactly 1. Now suppose that S is a dense subset in ℓ_∞. We need to prove that S is uncountable. Then for each element $x \in M$, there is a sequence in S converging to x, in particular, there is $s \in S$ with $d(x, s) < 1/2$. Let us choose such a point s for each $x \in M$. We cannot choose the same s for two distinct elements $x \neq y$ of M, because otherwise

$$1 = d(x, y) \leq d(x, s) + d(y, s) < 1/2 + 1/2 = 1.$$

Consequently, there is an injective map from M to S. Since M is uncountable, so is S.

8. We need to check that d' satisifies all four properties of a metric. The first three are clearly satisfied. To prove the fourth (triangle inequality), it suffices to show that for any nonnegative reals a, b, c with $a + b \geq c$ it holds that

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c},$$

which can be shown by straightforward calculation.

The second statement follows from the fact that for nonnegative reals a_n, the sequence a_n converges to 0 if and only if $a_n/(1 + a_n) \to 0$.

9.* Uniqueness is clear. Assume that there are two fixed points: $f(x) = x$ and $f(y) = y$. Since f is a contraction, $d(x, y) = d(f(x), f(y)) \leq q d(x, y)$ for some $0 < q < 1$. This is a contradiction unless $d(x, y) = 0 \Leftrightarrow x = y$.

To prove existence, pick an arbitrary point $x_0 \in X$ in our complete metric space. Let $x_1 = f(x_0)$; $x_2 = f(x_1)$; $x_3 = f(x_2)$ and so on. Our first goal is to show that (x_n) is a Cauchy sequence. We denote the distance $d(x_0, x_1)$ by r. Then

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \leq q d(x_0, x_1) = qr.$$

Similarly,

$$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq q d(x_1, x_2) \leq q^2 r.$$

After n steps we get that

$$d(x_n, x_{n+1}) \leq q^n r.$$

It follows that for $m > n$:

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \leq \left(q^n + q^{n+1} + \ldots + q^{m-1}\right) r \leq q^n \frac{r}{1-q}.$$

For any $\varepsilon > 0$, let N be a positive integer for which $q^N \frac{r}{1-q} < \varepsilon$. Then $d(x_n, x_m) < \varepsilon$ for any $n, m \geq N$. Thus (x_n) is indeed a Cauchy sequence.

Our metric space is complete, so (x_n) is convergent: $x_n \xrightarrow{d} x$ for some $x \in X$. We claim that x is a fixed point of f. Since $x_n \xrightarrow{d} x$, $d(x_n, x) \xrightarrow{d} 0$. However,

$$d\left(x_{n+1}, f(x)\right) = d\left(f(x_n), f(x)\right) \leq q d\left(x_n, x\right) \xrightarrow{d} 0.$$

It means that the sequence x_n converges both to x and to $f(x)$. By Exercise 3, it follows that $x = f(x)$.