Functional Analysis, BSM, Spring 2012

Homework set, Week 1

Solutions

1. If $Lx = \lambda x$ for some nonzero $x = (\alpha_1, \alpha_2, \ldots)$, then we have $x = (\alpha_1, \lambda\alpha_1, \lambda^2\alpha_1, \lambda^3\alpha_1, \ldots)$ with $\alpha_1 \neq 0$. So we need to determine the set of those λ for which $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ is in the given spaces $\mathbb{C}^{\mathbb{N}}$, ℓ_∞ and ℓ_p:
a) $\mathbb{C}$;
b) $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$;
c) $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

2. Clearly, the kernel is the set of constant functions and the range is the whole space $C^\infty[0,1]$. The set of eigenvalues is $\mathbb{R}$, since for any $\lambda \in \mathbb{R}$ the function $f(x) = e^{\lambda x}$ is an eigenvector with eigenvalue λ:

$$(Df)(x) = f'(x) = \lambda e^{\lambda x} = \lambda f(x).$$

3. Our assumption is that

$$\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty \text{ and } \sum_{i=1}^{\infty} |\beta_i|^2 < \infty;$$

we need to prove that

$$\sum_{i=1}^{\infty} |\alpha_i + \beta_i|^2 < \infty.$$

Using the triangle inequality and the fact that $2ab \leq a^2 + b^2$ for nonnegative reals a, b:

$$|\alpha_i + \beta_i|^2 \leq (|\alpha_i| + |\beta_i|)^2 = |\alpha_i|^2 + |\beta_i|^2 + 2|\alpha_i||\beta_i| \leq 2|\alpha_i|^2 + 2|\beta_i|^2.$$

It follows that

$$\sum_{i=1}^{\infty} |\alpha_i + \beta_i|^2 \leq 2\sum_{i=1}^{\infty} |\alpha_i|^2 + 2\sum_{i=1}^{\infty} |\beta_i|^2 < \infty.$$

One can prove the same statement for any ℓ_p, $p \geq 1$ space instead of ℓ_2. For nonnegative real numbers a and b it holds that

$$\sqrt[p]{\frac{a^p + b^p}{2}} \geq \frac{a+b}{2}.$$

This implies that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Consequently,

$$\sum_{i=1}^{\infty} |\alpha_i + \beta_i|^p \leq \sum_{i=1}^{\infty} (|\alpha_i| + |\beta_i|)^p \leq 2^{p-1}\sum_{i=1}^{\infty} |\alpha_i|^p + 2^{p-1}\sum_{i=1}^{\infty} |\beta_i|^p < \infty.$$

4. Consider the map T that takes

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \ldots)$$

to

$$(\alpha_3, \alpha_1, \alpha_5, \alpha_2, \alpha_7, \alpha_4, \alpha_9, \alpha_6, \alpha_{11}, \ldots).$$

It is not hard to check that T is a linear transformation from ℓ_∞ to itself. Clearly, 0 is not an eigenvalue for T. Suppose that some $\lambda \neq 0$ is an eigenvalue. The corresponding eigenvector must be

$$(\alpha_1, \lambda^{-1}\alpha_1, \lambda^1\alpha_1, \lambda^{-2}\alpha_1, \lambda^2\alpha_1, \lambda^{-3}\alpha_1, \lambda^3\alpha_1, \ldots)$$

for some $\alpha_1 \in \mathbb{C} \setminus \{0\}$. If $|\lambda| < 1$ or $|\lambda| > 1$, then the above sequence is clearly not bounded. So in these cases none of the possible eigenvectors are in ℓ_∞. However, if $|\lambda| = 1$, then the above sequence is bounded and it is indeed an eigenvector for T with eigenvalue λ.

5. Let

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 - 1/(4n) \\ 2nx - n + 1/2 & \text{if } 1/2 - 1/(4n) < x < 1/2 + 1/(4n) \\ 1 & \text{if } 1/2 + 1/(4n) \le x \le 1 \end{cases}$$

It is easy to check that (f_n) is a Cauchy sequence in $(C[0,1], d)$. However, it is not convergent. A heuristic proof: the sequence *should converge* to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 \\ 1 & \text{if } 1/2 < x \le 1 \end{cases}$$

but this is not a continuous function.
A rigorous proof: suppose that (f_n) converges to a continuous function f. We distinguish two cases. First, assume that $f(1/2) \ge 1/2$. Since f is continuous at $x = 1/2$, there exists a $\delta > 0$ such that $f(x) \ge 1/4$ for $x \in [1/2 - \delta, 1/2 + \delta]$. It follows that $|f_n(x) - f(x)| = |f(x)| \ge 1/4$ for $x \in [1/2 - \delta, 1/2 - 1/(4n)]$. Then

$$d(f_n, f) = \int_0^1 |f_n(x) - f(x)| \, dx \ge \int_{1/2-\delta}^{1/2-1/(4n)} |f_n(x) - f(x)| \, dx \ge \left(\delta - \frac{1}{4n}\right)\frac{1}{4},$$

which does not tend to 0 as $n \to \infty$.

If $f(1/2) \le 1/2$, then there exists a $\delta > 0$ such that $f(x) \le 3/4$ for $x \in [1/2 - \delta, 1/2 + \delta]$. It implies that $|f_n(x) - f(x)| = |1 - f(x)| \ge 1/4$ for $x \in [1/2 + 1/(4n), 1/2 + \delta]$. It follows (similarly as in the first case) that $d(f_n, f) \not\to 0$.

6.

$$\begin{aligned} |a_n - a_m| = |d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \le \\ &|d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \le d(y_n, y_m) + d(x_n, x_m). \end{aligned}$$

Let $\varepsilon > 0$. Since (x_n) is Cauchy, there is an N_1 such that $d(x_n, x_m) < \varepsilon/2$ for $n, m \ge N_1$. Similarly, there is an N_2 such that $d(y_n, y_m) < \varepsilon/2$ for $n, m \ge N_2$. Set $N = \max(N_1, N_2)$. Consequently, if $n, m \ge N$, then

$$|a_n - a_m| \le d(y_n, y_m) + d(x_n, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

7. Suppose that $x_1, x_2, \ldots \in X$ is a Cauchy sequence. We need to prove that it is convergent. Let

$$x_n = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots)$$

and $\varepsilon = 1/k$ for some fixed positive integer k. Since (x_n) is Cauchy, there is an N such that $d(x_n, x_m) < \varepsilon = 1/k$ for $n, m \ge N$. It follows that for $n, m \ge N$ we have

$$a_1^{(n)} = a_1^{(m)}; a_2^{(n)} = a_2^{(m)}; \ldots; a_k^{(n)} = a_k^{(m)}.$$

In other words, the first k elements of the sequences $x_N, x_{N+1}, \ldots$ are the same.

We showed that for any k, after a while ($n \ge N$) the k-th elements are the same in all x_n's. Let b_k denote this *stabilized* k-th element. Let y denote the 0-1 sequence $(b_1, b_2, \ldots)$. It is clear that $x_n \xrightarrow{d} y$.

8. We need to prove that for arbitrary $x \in X_1 \setminus \{0\}$:

$$\frac{\|STx\|_3}{\|x\|_1} \le \|S\|_{2,3}\|T\|_{1,2}.$$

If $Tx = 0$, then $STx = 0$, and we are done. Otherwise:

$$\frac{\|STx\|_3}{\|x\|_1} = \frac{\|STx\|_3}{\|Tx\|_2} \cdot \frac{\|Tx\|_2}{\|x\|_1} \le \|S\|_{2,3}\|T\|_{1,2}.$$