Functional Analysis, BSM, Spring 2012

Homework set, Week 1

1. What is the set of eigenvalues for the left shift operator

$$L : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_2, \alpha_3, \alpha_4, \ldots)$$

a) as a $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ operator;
b) as an $\ell_\infty \to \ell_\infty$ operator;
c) as an $\ell_p \to \ell_p$ operator?

2. Let

$$C^\infty[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is infinitely differentiable}\}$$

and D denote the differentiation operator on $C^\infty[0,1]$:

$$Df = f' \text{ for all } f \in C^\infty[0,1].$$

Determine the kernel $\ker D$, the range $\operatorname{ran} D$ and the set of eigenvalues for D.

3. W1P1. (5 points) Prove that if $(\alpha_1, \alpha_2, \ldots) \in \ell_2$ and $(\beta_1, \beta_2, \ldots) \in \ell_2$, then $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots) \in \ell_2$.

4. W1P2. (10 points) Show an operator (linear transformation) $T : \ell_\infty \to \ell_\infty$ such that the set of eigenvalues for T is $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

5. W1P3. (8 points) Consider $C[0,1]$, the set of continuous real functions on $[0,1]$, with the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x.$$

Show that $(C[0,1], d)$ is not complete.

6. W1P4. (10 points) Let (X,d) be a metric space. Suppose that $x_1, x_2, \ldots \in X$ and $y_1, y_2, \ldots \in X$ are Cauchy sequences. Set $a_n = d(x_n, y_n)$. Prove that $a_1, a_2, \ldots \in \mathbb{R}$ is a Cauchy sequence. (Since $\mathbb{R}$ is complete, this means that the sequence (a_n) is convergent.)

7. W1P5. (10 points) Consider the set of infinite 0-1 sequences

$$X = \{(a_1, a_2, \ldots) : a_i \in \{0,1\}\}$$

with the following metric:

$$d\left((a_1, a_2, \ldots), (b_1, b_2, \ldots)\right) = 1/k,$$

where k is the smallest positive integer for which $a_k \neq b_k$. (If there is no such k, that is, $a_i = b_i$ for each i, then the two sequences are the same. In that case, let their distance be 0.) Prove that (X,d) is a complete metric space.

8. W2P1. (5 points) Let $(X_1, \|\cdot\|_1)$; $(X_2, \|\cdot\|_2)$; $(X_3, \|\cdot\|_3)$ be normed spaces and $T \in B(X_1, X_2)$; $S \in B(X_2, X_3)$ bounded operators. Prove that

$$\|ST\|_{1,3} \leq \|S\|_{2,3} \|T\|_{1,2}.$$

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`