χ-bounded graph classes - results and problems

András Gyárfás

Alfréd Rényi Institute of Mathematics

2016 September, Bedlewo

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**
- if $\mathcal{G}$ is χ-bounded, what is its smallest χ-bounding function ?

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**
- if $\mathcal{G}$ is χ-bounded, what is its smallest χ-bounding function ?
- the class χ-bounded with the identity function $f(x) = x$ is the family of perfect graphs.

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**
- if $\mathcal{G}$ is χ-bounded, what is its smallest χ-bounding function ?
- the class χ-bounded with the identity function $f(x) = x$ is the family of perfect graphs.
- since $\omega(G) = 1$ implies $\chi(G) = 1$ and $\omega(G) \leq \chi(G)$, we may assume that $f(1) = 1, f(x) \geq x$ for any χ-bounding function f

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**
- if $\mathcal{G}$ is χ-bounded, what is its smallest χ-bounding function ?
- the class χ-bounded with the identity function $f(x) = x$ is the family of perfect graphs.
- since $\omega(G) = 1$ implies $\chi(G) = 1$ and $\omega(G) \leq \chi(G)$, we may assume that $f(1) = 1, f(x) \geq x$ for any χ-bounding function f
- the real test case is the class of graphs with $\omega(G) = 2$: a class $\mathcal{G}$ of graphs is **weakly** χ-**bounded** if there is a bound for the chromatic number of every triangle-free member of the class.

Terminology - Pokrzywna, Sept. 16-20, 1985

- a class $\mathcal{G}$ of graphs is χ-**bounded** if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$; such an f is called a χ-bounding function of $\mathcal{G}$. We consider classes that are **closed for induced subgraphs.**
- if $\mathcal{G}$ is χ-bounded, what is its smallest χ-bounding function ?
- the class χ-bounded with the identity function $f(x) = x$ is the family of perfect graphs.
- since $\omega(G) = 1$ implies $\chi(G) = 1$ and $\omega(G) \leq \chi(G)$, we may assume that $f(1) = 1, f(x) \geq x$ for any χ-bounding function f
- the real test case is the class of graphs with $\omega(G) = 2$: a class $\mathcal{G}$ of graphs is **weakly** χ**-bounded** if there is a bound for the chromatic number of every triangle-free member of the class.
- **metaconjecture:** „natural" weakly χ-bounded graph classes are χ-bounded

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Let $Forb(\mathcal{H})$ denote the class of graphs with no induced subgraph from $\mathcal{H}$.

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Let $Forb(\mathcal{H})$ denote the class of graphs with no induced subgraph from $\mathcal{H}$.

- $Forb(T)$ is weakly χ-bounded if T is a radius two tree (Gy. Szemerédi, Tuza, 1980)

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Let $Forb(\mathcal{H})$ denote the class of graphs with no induced subgraph from $\mathcal{H}$.

- $Forb(T)$ is weakly χ-bounded if T is a radius two tree (Gy. Szemerédi, Tuza, 1980)
- $Forb(B)$ is χ-bounded if B is a broom (Gy. 1985)

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Let $Forb(\mathcal{H})$ denote the class of graphs with no induced subgraph from $\mathcal{H}$.

- $Forb(T)$ is weakly χ-bounded if T is a radius two tree (Gy. Szemerédi, Tuza, 1980)
- $Forb(B)$ is χ-bounded if B is a broom (Gy. 1985)
- $Forb(T)$ is χ-bounded if T is a radius two tree (Kierstead, Penrice, 1994)

Forbidding trees

If a χ-bounded graph class is defined by one forbidden (induced) subgraph H then H must be acyclic. The reverse statement is a long standing conjecture.

Conjecture

(Forbidden tree conjecture, Gy. 1975, Sumner 1981) For any acyclic graph H, the class of graphs without (induced) H is χ-bounded.

Let $Forb(\mathcal{H})$ denote the class of graphs with no induced subgraph from $\mathcal{H}$.

- $Forb(T)$ is weakly χ-bounded if T is a radius two tree (Gy. Szemerédi, Tuza, 1980)
- $Forb(B)$ is χ-bounded if B is a broom (Gy. 1985)
- $Forb(T)$ is χ-bounded if T is a radius two tree (Kierstead, Penrice, 1994)
- $Forb(T)$ is χ-bounded if T is a subdivided star (Scott, 1997)

Forbidding the claw

- the best bounding function $f(x)$ satisfies $\frac{R(3,x+1)-1}{2} \leq f(x) \leq R(3,x)$ (easy)

Forbidding the claw

- the best bounding function $f(x)$ satisfies $\frac{R(3,x+1)-1}{2} \leq f(x) \leq R(3,x)$ (easy)
- improvement: $f(x) \leq \frac{R(3,x)+x}{2}$ (King, 2009), this gives

Forbidding the claw

- the best bounding function $f(x)$ satisfies $\frac{R(3,x+1)-1}{2} \leq f(x) \leq R(3,x)$ (easy)
- improvement: $f(x) \leq \frac{R(3,x)+x}{2}$ (King, 2009), this gives
- $f(2) = 3, 4 \leq f(3) \leq 5, f(4) = 7, 9 \leq f(5) \leq 10, 11 \leq f(6) \leq 13$

Forbidding the claw

- the best bounding function $f(x)$ satisfies $\frac{R(3,x+1)-1}{2} \leq f(x) \leq R(3,x)$ (easy)
- improvement: $f(x) \leq \frac{R(3,x)+x}{2}$ (King, 2009), this gives
- $f(2) = 3, 4 \leq f(3) \leq 5, f(4) = 7, 9 \leq f(5) \leq 10, 11 \leq f(6) \leq 13$
- surprise: the class of connected graphs with $\alpha(G) \geq 3$ is χ-bounded with $2x$ (Chudnovsky, Seymour, 2010)

Forbidding other small forests

Best χ-bounding functions of $Forb(H)$

- $H = P_4 : f(x) = x$ (perfect Seinsche, 1974)

Forbidding other small forests

Best χ-bounding functions of $Forb(H)$

- $H = P_4 : f(x) = x$ (perfect Seinsche, 1974)
- $H = P_3 \cup K_1$: asymptotic to $\frac{R(3,x+1)-1}{2}$

Forbidding other small forests

Best χ-bounding functions of $Forb(H)$

- $H = P_4 : f(x) = x$ (perfect Seinsche, 1974)
- $H = P_3 \cup K_1$: asymptotic to $\frac{R(3,x+1)-1}{2}$
- $H = K_2 \cup 2K_1$: $\frac{R(3,x+1)-1}{2} \leq f(x) \leq \binom{x+1}{2}$ (Randerath, Schiermeyer, 2004)

Forbidding other small forests

Best χ-bounding functions of $Forb(H)$

- $H = P_4 : f(x) = x$ (perfect Seinsche, 1974)
- $H = P_3 \cup K_1$: asymptotic to $\frac{R(3,x+1)-1}{2}$
- $H = K_2 \cup 2K_1$: $\frac{R(3,x+1)-1}{2} \leq f(x) \leq \binom{x+1}{2}$ (Randerath, Schiermeyer, 2004)
- $H = 2K_2 : \frac{R(C_4,K_{x+1})}{3}) \leq f(x) \leq \binom{x+1}{2}$ (Wagon, 1980)

Forbidding a tree and something else

- For any tree T, $Forb(T, K_{2,2})$ is weakly χ-bounded, in fact any graph G from the class satisfies $\chi(G) \leq |V(T)|$ (Gy. Szemerédi, Tuza, 1980)

Forbidding a tree and something else

- For any tree T, $Forb(T, K_{2,2})$ is weakly χ-bounded, in fact any graph G from the class satisfies $\chi(G) \leq |V(T)|$ (Gy. Szemerédi, Tuza, 1980)
- For any tree T and any $t \geq 2$, $Forb(T, K_{t,t})$ is χ-bounded (Kiersted, Rödl, 1996)

Forbidding sets of induced cycles

In a series of papers some of my old conjectures have been proved:

Forbidding sets of induced cycles

In a series of papers some of my old conjectures have been proved:

- 1. The class of graphs without odd holes is χ-bounded (with bounding function 2^{3^x}, Scott and Seymour 2014). The subclass with $\omega(G) \leq 3$ is at most 4-chromatic (Chudnovsky, Robertson, Seymour, Thomas, 2010)

Forbidding sets of induced cycles

In a series of papers some of my old conjectures have been proved:

- 1. The class of graphs without odd holes is χ-bounded (with bounding function 2^{3^x}, Scott and Seymour 2014). The subclass with $\omega(G) \leq 3$ is at most 4-chromatic (Chudnovsky, Robertson, Seymour, Thomas, 2010)
- 2. graphs without long holes are χ-bounded (Chudnovsky, Scott and Seymour 2015)

Forbidding sets of induced cycles

In a series of papers some of my old conjectures have been proved:

- 1. The class of graphs without odd holes is χ-bounded (with bounding function 2^{3^x}, Scott and Seymour 2014). The subclass with $\omega(G) \leq 3$ is at most 4-chromatic (Chudnovsky, Robertson, Seymour, Thomas, 2010)
- 2. graphs without long holes are χ-bounded (Chudnovsky, Scott and Seymour 2015)

My third conjecture, whether graphs without long odd holes are χ-bounded still holds but probably not for long...

- 3. For any fixed $t > 0$, graphs without t consecutive holes are *weakly* χ-bounded (Scott and Seymour 2015)

Forbidding sets of induced cycles

In a series of papers some of my old conjectures have been proved:

- 1. The class of graphs without odd holes is χ-bounded (with bounding function 2^{3^x}, Scott and Seymour 2014). The subclass with $\omega(G) \leq 3$ is at most 4-chromatic (Chudnovsky, Robertson, Seymour, Thomas, 2010)
- 2. graphs without long holes are χ-bounded (Chudnovsky, Scott and Seymour 2015)

My third conjecture, whether graphs without long odd holes are χ-bounded still holds but probably not for long...

- 3. For any fixed $t > 0$, graphs without t consecutive holes are *weakly* χ-bounded (Scott and Seymour 2015)
- 4. The class without induced cycles of length $0 \pmod 3$ (Bonamy, Charbit, Thomassé, 2014)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: $Forb^*(H)$ is χ-bounded for any H (Scott, 1997)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: $Forb^*(H)$ is χ-bounded for any H (Scott, 1997)
- true if H is a paw or a bull or a necklace (Chudnovsky, Penev, Scott, Trotignon, 2010)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: $Forb^*(H)$ is χ-bounded for any H (Scott, 1997)
- true if H is a paw or a bull or a necklace (Chudnovsky, Penev, Scott, Trotignon, 2010)
- true for maximal triangle free H (Bousquet, Thomassé, 2011)

Forbidding induced topological subgraphs

Let *Forb*$^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, *Forb*$^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: *Forb*$^*(H)$ is χ-bounded for any H (Scott, 1997)
- true if H is a paw or a bull or a necklace (Chudnovsky, Penev, Scott, Trotignon, 2010)
- true for maximal triangle free H (Bousquet, Thomassé, 2011)
- Not true in general: consequence of the surprising discovery that intersection graphs of straight line segments of the plane is not a χ-bounded class (Pawlik, Kozik, Krawczyk,Lasoń, Micek, Trotter, Walczak, 2012)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: $Forb^*(H)$ is χ-bounded for any H (Scott, 1997)
- true if H is a paw or a bull or a necklace (Chudnovsky, Penev, Scott, Trotignon, 2010)
- true for maximal triangle free H (Bousquet, Thomassé, 2011)
- Not true in general: consequence of the surprising discovery that intersection graphs of straight line segments of the plane is not a χ-bounded class (Pawlik, Kozik, Krawczyk,Lasoń, Micek, Trotter, Walczak, 2012)
- More counterexamples (Chalopin, Esperet, Li, Ossona de Mendez, 2014)

Forbidding induced topological subgraphs

Let $Forb^*(H)$ denote the class of graphs that do not contain any subdivision of H as an induced subgraph.

- For any tree T, $Forb^*(T)$ is χ-bounded (Scott, 1997)
- Conjecture: $Forb^*(H)$ is χ-bounded for any H (Scott, 1997)
- true if H is a paw or a bull or a necklace (Chudnovsky, Penev, Scott, Trotignon, 2010)
- true for maximal triangle free H (Bousquet, Thomassé, 2011)
- Not true in general: consequence of the surprising discovery that intersection graphs of straight line segments of the plane is not a χ-bounded class (Pawlik, Kozik, Krawczyk,Lasoń, Micek, Trotter, Walczak, 2012)
- More counterexamples (Chalopin, Esperet, Li, Ossona de Mendez, 2014)
- Conjecture: true when H is a forest of chandeliers (Chudnovsky, Scott, Seymour, 2015)

Forbidding other families of graphs - I.

Conjecture

(Aboulker, Bousquet, 2015) For any fixed k the class $\mathcal{C}_k$ of graphs that do not contain cycles with exactly k chords is χ-bounded.

Forbidding other families of graphs - I.

Conjecture

(Aboulker, Bousquet, 2015) For any fixed k the class $\mathcal{C}_k$ of graphs that do not contain cycles with exactly k chords is χ-bounded.

- $\mathcal{C}_1$ is χ-bounded by $f(x) = max\{3, x\}$ (Trotignon, Vušković, 2009).

Forbidding other families of graphs - I.

Conjecture

(Aboulker, Bousquet, 2015) For any fixed k the class $\mathcal{C}_k$ of graphs that do not contain cycles with exactly k chords is χ-bounded.

- $\mathcal{C}_1$ is χ-bounded by $f(x) = max\{3, x\}$ (Trotignon, Vušković, 2009).
- $\mathcal{C}_2$ is χ-bounded by $f(x) = 6$, $\mathcal{C}_3$ is χ-bounded by $f(x) = max\{96, x + 1\}$ (Aboulker, Bousquet, 2015).

Forbidding other families of graphs - I.

Conjecture

(Aboulker, Bousquet, 2015) For any fixed k the class $\mathcal{C}_k$ of graphs that do not contain cycles with exactly k chords is χ-bounded.

- $\mathcal{C}_1$ is χ-bounded by $f(x) = max\{3, x\}$ (Trotignon, Vušković, 2009).
- $\mathcal{C}_2$ is χ-bounded by $f(x) = 6$, $\mathcal{C}_3$ is χ-bounded by $f(x) = max\{96, x + 1\}$ (Aboulker, Bousquet, 2015).

Conjecture

(Geelen, 1995) For every graph H, the class of graphs without a vertex-minor H is χ-bounded. (Vertex-minor: can be obtained by vertex removal and complementing neighborhoods.)

Forbidding other families of graphs - I.

Conjecture

(Aboulker, Bousquet, 2015) For any fixed k the class $\mathcal{C}_k$ of graphs that do not contain cycles with exactly k chords is χ-bounded.

- $\mathcal{C}_1$ is χ-bounded by $f(x) = max\{3, x\}$ (Trotignon, Vušković, 2009).
- $\mathcal{C}_2$ is χ-bounded by $f(x) = 6$, $\mathcal{C}_3$ is χ-bounded by $f(x) = max\{96, x + 1\}$ (Aboulker, Bousquet, 2015).

Conjecture

(Geelen, 1995) For every graph H, the class of graphs without a vertex-minor H is χ-bounded. (Vertex-minor: can be obtained by vertex removal and complementing neighborhoods.)

Geelen's conjecture is proved if H is path with an additional vertex connected to all vertices of the path (Choi, Kwon, Oum, 2015)

Forbidding other families of graphs - II.

A k-wheel is an induced cycle and one additional vertex connected to at least k vertices of the cycle. A wheel is a 3-wheel.

Conjecture

The class of wheel-free graphs is χ-bounded (Trotignon, 2010)

Forbidding other families of graphs - II.

A k-wheel is an induced cycle and one additional vertex connected to at least k vertices of the cycle. A wheel is a 3-wheel.

Conjecture

The class of wheel-free graphs is χ-bounded (Trotignon, 2010)

- For fixed k, ℓ, the class $Forb(\mathrm{k}-wheel, K_{\ell,\ell})$ is weakly χ-bounded (Bousquet, Thomassé, 2015)

Forbidding other families of graphs - II.

A k-wheel is an induced cycle and one additional vertex connected to at least k vertices of the cycle. A wheel is a 3-wheel.

Conjecture

The class of wheel-free graphs is χ-bounded (Trotignon, 2010)

- For fixed k, ℓ, the class $Forb(\text{k}-wheel, K_{\ell,\ell})$ is weakly χ-bounded (Bousquet, Thomassé, 2015)
- For fixed ℓ, the class $Forb(wheel, K_{\ell,\ell})$ is χ-bounded (Bousquet, Thomassé, 2015)

Directed graphs

Let D be a digraph and $Forb(D)$ the class of digraphs that do not contain an induced copy of D. The χ-boundedness of the classes $Forb(D)$ behave quite differently even for very simple D. For example a path on four vertices can be oriented in three essentially different way.

Directed graphs

Let D be a digraph and $Forb(D)$ the class of digraphs that do not contain an induced copy of D. The χ-boundedness of the classes $Forb(D)$ behave quite differently even for very simple D. For example a path on four vertices can be oriented in three essentially different way.

- For $D_1 = (1,2),(2,3)(4,3)$, $Forb(D_1)$ is perfect (Chvatal, 1989)

Directed graphs

Let D be a digraph and $Forb(D)$ the class of digraphs that do not contain an induced copy of D. The χ-boundedness of the classes $Forb(D)$ behave quite differently even for very simple D. For example a path on four vertices can be oriented in three essentially different way.

- For $D_1 = (1,2),(2,3)(4,3)$, $Forb(D_1)$ is perfect (Chvatal, 1989)
- For $D_2 = (1,2),(3,2),(3,4)$, $Forb(D_2)$ is not χ-bounded

Directed graphs

Let D be a digraph and $Forb(D)$ the class of digraphs that do not contain an induced copy of D. The χ-boundedness of the classes $Forb(D)$ behave quite differently even for very simple D. For example a path on four vertices can be oriented in three essentially different way.

- For $D_1 = (1,2),(2,3)(4,3)$, $Forb(D_1)$ is perfect (Chvatal, 1989)
- For $D_2 = (1,2),(3,2),(3,4)$, $Forb(D_2)$ is not χ-bounded
- For $D_3 = (1,2),(2,3),(3,4)$, $Forb(D_3)$ is not χ-bounded (Kierstead, Trotter, 1992)

Directed graphs

Let D be a digraph and $Forb(D)$ the class of digraphs that do not contain an induced copy of D. The χ-boundedness of the classes $Forb(D)$ behave quite differently even for very simple D. For example a path on four vertices can be oriented in three essentially different way.

- For $D_1 = (1,2),(2,3)(4,3)$, $Forb(D_1)$ is perfect (Chvatal, 1989)
- For $D_2 = (1,2),(3,2),(3,4)$, $Forb(D_2)$ is not χ-bounded
- For $D_3 = (1,2),(2,3),(3,4)$, $Forb(D_3)$ is not χ-bounded (Kierstead, Trotter, 1992)
- Is $Forb(S_{s,t})$ χ-bounded, where $S_{p,q}$ is the star with s outgoing and t incoming edges? (Kierstead, Rödl, 1996 and Aboulker et. al. 2016)

Preserving χ-boundedness

It is useful to have operations preserving χ-boundedness

Preserving χ-boundedness

It is useful to have operations preserving χ-boundedness

- gluing along small rank cuts; Dvořak, Král, 2012

Preserving χ-boundedness

It is useful to have operations preserving χ-boundedness

- gluing along small rank cuts; Dvořak, Král, 2012
- substitutions; Chudnovsky, Penev, Scott, Trotignon, 2013

Preserving χ-boundedness

It is useful to have operations preserving χ-boundedness

- gluing along small rank cuts; Dvořak, Král, 2012
- substitutions; Chudnovsky, Penev, Scott, Trotignon, 2013
- gluing (along cliques and bounded number of vertices) ; Chudnovsky, Penev, Scott, Trotignon, 2013

Preserving χ-boundedness

It is useful to have operations preserving χ-boundedness

- gluing along small rank cuts; Dvořak, Král, 2012
- substitutions; Chudnovsky, Penev, Scott, Trotignon, 2013
- gluing (along cliques and bounded number of vertices) ; Chudnovsky, Penev, Scott, Trotignon, 2013
- gluing along homogeneous subsets and amalgams; Penev, 2014

Complementary χ-bounding functions

- A function f is complementary bounded if for any graph family $\mathcal{G}$ with bounding function f, the family $\{G^c : G \in calG\}$ is also χ-bounded (with some bounding function g). The smallest such g is denoted by f^*.

Complementary χ-bounding functions

- A function f is complementary bounded if for any graph family $\mathcal{G}$ with bounding function f, the family $\{G^c : G \in calG\}$ is also χ-bounded (with some bounding function g). The smallest such g is denoted by f^*.
- Weak perfect graph theorem (Lovász): $f(x) = x$ is a self-complementary bounding function

Complementary χ-bounding functions

- A function f is complementary bounded if for any graph family $\mathcal{G}$ with bounding function f, the family $\{G^c : G \in calG\}$ is also χ-bounded (with some bounding function g). The smallest such g is denoted by f^*.
- Weak perfect graph theorem (Lovász): $f(x) = x$ is a self-complementary bounding function
- (Gy. 1985) If f is complementary bounded then $f(x) = o(x)$

Complementary χ-bounding functions

- A function f is complementary bounded if for any graph family $\mathcal{G}$ with bounding function f, the family $\{G^c : G \in calG\}$ is also χ-bounded (with some bounding function g). The smallest such g is denoted by f^*.
- Weak perfect graph theorem (Lovász): $f(x) = x$ is a self-complementary bounding function
- (Gy. 1985) If f is complementary bounded then $f(x) = o(x)$
- (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013) The function $f(x) = x + x/\log^j(x)$ is not complementary bounded for any fixed j

Complementary χ-bounding functions

- A function f is complementary bounded if for any graph family $\mathcal{G}$ with bounding function f, the family $\{G^c : G \in calG\}$ is also χ-bounded (with some bounding function g). The smallest such g is denoted by f^*.
- Weak perfect graph theorem (Lovász): $f(x) = x$ is a self-complementary bounding function
- (Gy. 1985) If f is complementary bounded then $f(x) = o(x)$
- (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013) The function $f(x) = x + x/\log^j(x)$ is not complementary bounded for any fixed j
- Conjecture (Gy. 1985) The function $f(x) = x + c$ is complementary for fixed c

Complements of eventually identity functions

Let F_c denote the set of functions with $f(x) = x$ for $x \geq c$. The following results and problems are from (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013).

Complements of eventually identity functions

Let F_c denote the set of functions with $f(x) = x$ for $x \geq c$. The following results and problems are from (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013).

Theorem

For all c and $f \in F_c$, f is complementary bounded.

Complements of eventually identity functions

Let F_c denote the set of functions with $f(x) = x$ for $x \geq c$. The following results and problems are from (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013).

Theorem

For all c and $f \in F_c$, f is complementary bounded.

Conjecture

The function

$$g(x) = \left\{ \begin{array}{ll} x & \text{if } x \geq 3 \\ 3 & \text{if } x = 2 \end{array} \right\}. \tag{1}$$

has best complementary bounding function is $g^* = \lfloor \frac{8x}{5} \rfloor$.

Complements of eventually identity functions

Let F_c denote the set of functions with $f(x) = x$ for $x \geq c$. The following results and problems are from (Gy., Li, Machado, Sebő, Thomassé, Trotignon, 2013).

Theorem

For all c and $f \in F_c$, f is complementary bounded.

Conjecture

The function

$$g(x) = \left\{ \begin{array}{ll} x & \text{if } x \geq 3 \\ 3 & \text{if } x = 2 \end{array} \right\}. \quad (1)$$

has best complementary bounding function is $g^* = \lfloor \frac{8x}{5} \rfloor$.

This conjecture is true for 3-colorable graphs (in fact for every graph whose induced subgraphs H satisfy $\alpha(H) \geq \frac{|V(H)|}{3}$).

χ-bounded classes in geometry

χ-boundedness of geometric objects is defined through their intersection graphs. I just mention d-dimensional boxes and straight line segments of the plane.

χ-bounded classes in geometry

χ-boundedness of geometric objects is defined through their intersection graphs. I just mention d-dimensional boxes and straight line segments of the plane.

- 1-dimensional boxes: perfect (Gallai's theorems on intervals)

χ-bounded classes in geometry

χ-boundedness of geometric objects is defined through their intersection graphs. I just mention d-dimensional boxes and straight line segments of the plane.

- 1-dimensional boxes: perfect (Gallai's theorems on intervals)
- 2-dimensional boxes: χ-bounded by quadratic function (Asplund, Grünbaum, 1960)

χ-bounded classes in geometry

χ-boundedness of geometric objects is defined through their intersection graphs. I just mention d-dimensional boxes and straight line segments of the plane.

- 1-dimensional boxes: perfect (Gallai's theorems on intervals)
- 2-dimensional boxes: χ-bounded by quadratic function (Asplund, Grünbaum, 1960)
- 3-dimensional boxes: (surprise) not (weakly) χ-bounded (Burling, 1965)

χ-bounded classes in geometry

χ-boundedness of geometric objects is defined through their intersection graphs. I just mention d-dimensional boxes and straight line segments of the plane.

- 1-dimensional boxes: perfect (Gallai's theorems on intervals)
- 2-dimensional boxes: χ-bounded by quadratic function (Asplund, Grünbaum, 1960)
- 3-dimensional boxes: (surprise) not (weakly) χ-bounded (Burling, 1965)
- straight-line segments: (surprise) not (weakly) χ-bounded (Pawlik et al. 2013)

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)
- the best on-line χ-bounding function for intervals is $3x - 2$ (Kierstead, Trotter, 1981)

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)
- the best on-line χ-bounding function for intervals is $3x - 2$ (Kierstead, Trotter, 1981)
- first fit coloring on intervals has linear χ-bounding function (Kierstead 1988), best bounds $5x, 8x$

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)
- the best on-line χ-bounding function for intervals is $3x - 2$ (Kierstead, Trotter, 1981)
- first fit coloring on intervals has linear χ-bounding function (Kierstead 1988), best bounds $5x, 8x$
- first fit is a perfect coloring on $Forb(P_4)$ but no on-line χ-bounding function exists on $Forb(P_6)$ (Gy., Lehel, 1988)

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)
- the best on-line χ-bounding function for intervals is $3x - 2$ (Kierstead, Trotter, 1981)
- first fit coloring on intervals has linear χ-bounding function (Kierstead 1988), best bounds $5x, 8x$
- first fit is a perfect coloring on $Forb(P_4)$ but no on-line χ-bounding function exists on $Forb(P_6)$ (Gy., Lehel, 1988)
- there is χ-bounded on-line coloring on $Forb(P_5)$ (Gy., Lehel, 1991)

Algorithms to find χ-bounded colorings

I just mention intervals of the line and $Forb(P_k)$.

- On-line and off-line algorithms to find χ-bounded colorings
- intervals has off-line perfect colorings (Gallai)
- the best on-line χ-bounding function for intervals is $3x - 2$ (Kierstead, Trotter, 1981)
- first fit coloring on intervals has linear χ-bounding function (Kierstead 1988), best bounds $5x, 8x$
- first fit is a perfect coloring on $Forb(P_4)$ but no on-line χ-bounding function exists on $Forb(P_6)$ (Gy., Lehel, 1988)
- there is χ-bounded on-line coloring on $Forb(P_5)$ (Gy., Lehel, 1991)
- the first fit coloring on $Forb(P_5)$ is χ-bounded (Kierstead, Penrice, Trotter, 1994)

Two of my 1985 problems that seemingly nobody touched

Two of my 1985 problems that seemingly nobody touched

- 1. Estimate the best bounding function of the union of two perfect graphs

Two of my 1985 problems that seemingly nobody touched

- 1. Estimate the best bounding function of the union of two perfect graphs
- 2. Let $\mathcal{G}$ be the family of graphs whose induced subgraphs satisfy

$$\alpha(G)\omega(G) \geq |V(G)| - 1.$$

Is $\mathcal{G}$ χ-bounded?

My three favorites

My three favorites

- Let $\mathcal{G}$ be the family of graphs with χ-bounding function $f(x) = x + 1$. Is the complementary family $\{G : G^c \in \mathcal{G}\}$ χ-bounded? (Gy. 1985)

- Let $\mathcal{G}$ be the family of graphs with χ-bounding function $f(x) = x + 1$. Is the complementary family $\{G : G^c \in \mathcal{G}\}$ χ-bounded? (Gy. 1985)
- For any fixed $t > 0$, graphs without t consecutive holes are χ-bounded (Scott and Seymour 2015)

My three favorites

- Let $\mathcal{G}$ be the family of graphs with χ-bounding function $f(x) = x + 1$. Is the complementary family $\{G : G^c \in \mathcal{G}\}$ χ-bounded? (Gy. 1985)
- For any fixed $t > 0$, graphs without t consecutive holes are χ-bounded (Scott and Seymour 2015)
- Let $\mathcal{G}$ be the family of graphs in which every path induces an at most 3-chromatic subgraph. Is $\mathcal{G}$ χ-bounded? (A simplified version of an Erdős - Hajnal problem, Gy. 1997)

Thanks for the attention!