Large monochromatic components in edge colorings of graphs

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- Questions and conjectures

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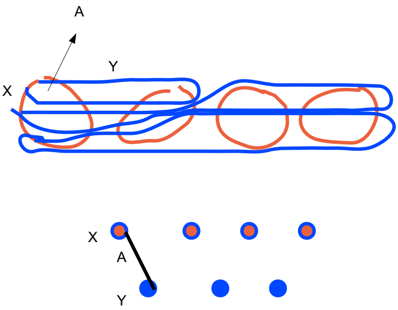
- *If two partitions are given on a ground set such that each pair of elements is covered by some block of the partitions then one of the partitions is trivial - i.e. has only one block.*
- *Pairwise intersecting edges of a bipartite multigraph has a common vertex.*

The dual of partitions



The dual of two partitions is a bipartite multigraph

Type of spanning trees

Height two tree

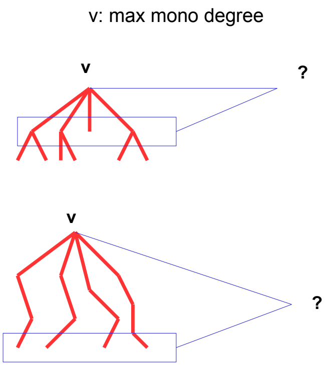
Octopus

Broom

Height two, octopus

Theorem

(Bialostocki, Dierker, Voxman, 1992)In every 2-coloring of K_n there exists a monochromatic spanning octopus and also a monochromatic spanning tree of height at most two.

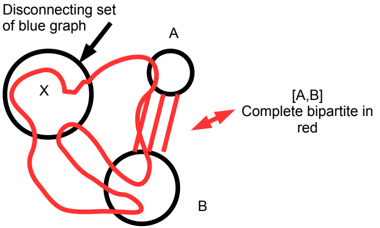


broom

Theorem

(Burr 1992) In every 2-coloring of K_n there exists a monochromatic spanning broom.

w.l.o.g the red graph is at least as connected as the blue graph



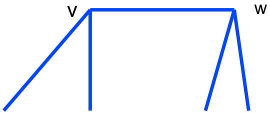
Union of a cycle and a complete bipartite graph has a spanning broom!

diameter three subgraph

Theorem

(Bialostocki 1992, Mubayi 2002, West 2000) In every 2-coloring of a complete graph there is a monochromatic spanning subgraph of diameter at most three.

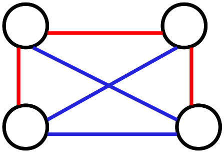
If v.w are at distance > 2 in the red graph then there is a blue spanning double star



Large mono diameter two subgraph

Theorem

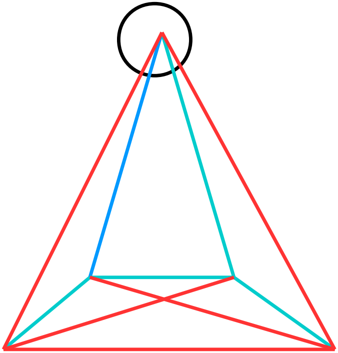
(Erdős - Fowler, 1999) In every 2-coloring of K_n there is a monochromatic subgraph of diameter at most two with at least $\frac{3n}{4}$ vertices. This is sharp as the following figure shows.



2-connected subgraphs

Theorem

(Bollobás-Gy. 2008) For $n \geq 5$ there is a monochromatic 2-connected subgraph with at least $n-2$ vertices in every 2-coloring of K_n. This is sharp as the following figure shows.

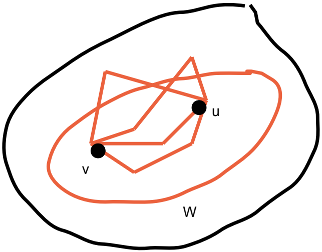


High connectivity plus small diameter

High connectivity plus small diameter

Lemma

(Gy-Sárközy-Szemerédi to appear) For every k and for every 2-colored K_n there exists $W \subset V(K_n)$ and a color such that $|W| \geq n - 28k$ and any two vertices in W can be connected in that color by k internally vertex disjoint paths, each with length at most three.



Gallai-colorings

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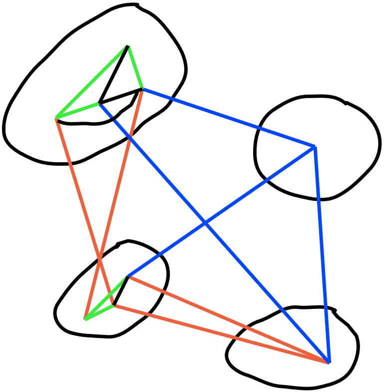
Theorem

(Gy. - Simonyi 2004) Every Gallai-coloring can be obtained from a 2-colored complete graph with at least two vertices by substituting Gallai colored complete graphs into its vertices.

Gallai-coloring by substitution



Substituting into red-blue coloring

Extending results for Gallai-colorings

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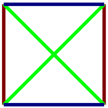
All - except 3. - can be obtained as 'black-box' extensions from the corresponding 2-coloring results...(Gy-Sárközy-Sebő-Selkow to appear)

Multicolorings from affine planes

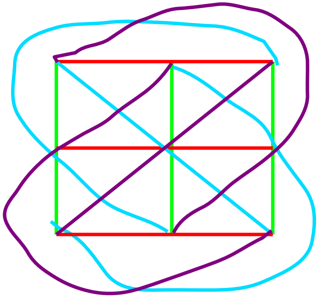
Consider an affine plane of order $r-1$ that is r partitions of a ground set $V, |V| = (r-1)^2$ into blocks of size $r-1$ so that each pair of elements of V is covered by a unique block. (If $r-1$ is a prime power, affine planes indeed exist.) There is a natural way to color the edges of a complete graph with vertex set V: for $i = 1, 2, \ldots, r$ color the pairs within the blocks of the i-th partition class with color i. For example, for $r = 3$ we obtain the 3-coloring of K_4 (a factorization), for $r = 4$ we obtain the 4-coloring of K_9 where each color class is the union of three vertex disjoint triangles.
The size of each monochromatic component is $\frac{1}{r-1}$ fraction of the total number of vertices.

Multicolorings - 3 and 4 colors

Colorings from affine planes

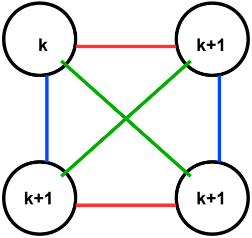


3-coloring



4-coloring

Substitution



Optimal substitution to get 3-coloring of a complete graph on 4k+3 vertices without monochromatic component of 2k+3 vertices

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Largest mono components in r-colorings

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The result can be proved by two different techniques.

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Lemma

(Mubayi 2002 and Liu-Morris-Prince 2004) In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic **double star** with at least $\frac{n}{r}$ vertices.

Large mono double stars in complete bipartite graphs

Proof. Suppose that $G = [A, B]$ is an r-colored complete bipartite graph, let $d_i(v)$ denote the degree of v in color i. For any edge ab of color i, $a \in A, b \in B$, set $c(a, b) = d_i(a) + d_i(b)$. Using the Cauchy-Schwartz inequality, we get

$$\sum_{ab\in E(G)} c(a,b) = \sum_{a\in A}\sum_{i=1}^{r} d_i^2(a) + \sum_{b\in B}\sum_{i=1}^{r} d_i^2(b) \geq |A|r\left(\frac{\sum_{a\in A}\sum_{i=1}^{r} d_i(a)}{|A|r}\right)^2$$

$$+|B|r\left(\frac{\sum_{b\in B}\sum_{i=1}^{r} d_i(b)}{|B|r}\right)^2 = |A||B|\left(\frac{|A|+|B|}{r}\right),$$

therefore for some $a \in A, b \in B$, $c(a,b) \geq \frac{|A|+|B|}{r}$ i.e. there is a monochromatic double star of the required size. □

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One gets more: either a mono spanning tree or a mono double star with at least $\frac{n}{r-1}$ vertices.

Fractional transversals - Füredi's method

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A *fractional matching* of a hypergraph is a non-negative weighting on the edges such that the sum of weights over the edges containing any fixed vertex is at most 1. The *value* of a fractional matching is the sum of the weights over all edges of the hypergraph. Then $\nu^*(\mathcal{H})$ is the maximum of the values over all fractional matchings of $\mathcal{H}$.

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By LP duality, $\tau^*(\mathcal{H}) = \nu^*(\mathcal{H})$ holds for every hypergraph $\mathcal{H}$.

Large mono component - second proof

Assume that the edges of K_n are r-colored. To find a monochromatic component with at least $\frac{n}{r-1}$ vertices is equivalent with finding a vertex of degree at least $\frac{n}{r-1}$ in an intersecting r-partite multihypergraph $\mathcal{H}$ with n edges.

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$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq r-1$$

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where D is the maximum degree of $\mathcal{H}$.
Thus we have

$$\frac{n}{r-1} = \frac{|E(\mathcal{H})|}{r-1} \leq D(\mathcal{H})$$

.

Local colorings

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(GY.-Sárközy 2008) $f(n, r) \geq \frac{rn}{r^2-r+1}$ *with equality if a finite plane of order* $r-1$ *exists and* $r^2 - r + 1$ *divides* n.

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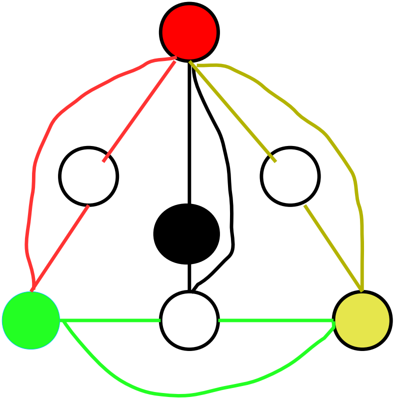
Both methods can be used to prove this theorem.

Optimal local 3-coloring with seven colors



Local 3-coloring (with seven colors)
each mono component has at most 3/7
of the total number of vertices

Hypergraph colorings

Parallel classes of hyperplanes in t-dimensional affine spaces give optimal colorings of t-uniform hypergraphs.

Theorem

(Füredi-Gy. 1991) In every r-coloring of the edges of the complete t-uniform hypergraph on n vertices, there is a connected monochromatic subhypergraph on at least $\frac{n}{q}$ vertices - where q is the smallest integer satisfying $r \leq \sum_{i=0}^{t-1} q^i$. The result is best possible if q is a prime power and n is divisible by q^t.

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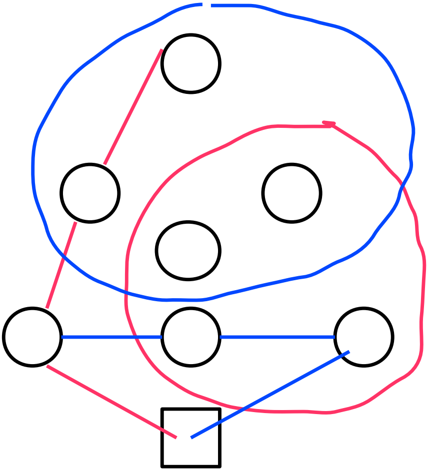
The lower bound comes by Füredi's method. The upper bound is coming by substitutions into affine spaces illustrated on the next slide.

Optimal 7-coloring of the complete 3-uniform hypergraph



7-coloring of a complete 3-uniform hypergraph based on the affine 3-dim space of order 2 – each color class has two equal components.

Bialostocki and Dierker conjectured - and proved (1990) for prime n - that in every coloring of the edges of K_{n+1} with colors in $\mathbb{Z}_n$ there is a spanning tree with color sum zero.

Coloring by group elements - zero sum spanning trees

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The general conjecture was proved by Füredi-Kleitman (1992) and by Schriver-Seymour (1991)

Coloring geometric graphs - non-crossing spanning trees

Coloring geometric graphs - non-crossing spanning trees

A *geometric graph* is a graph whose vertices are in the plane in general position and whose edges are straight-line segments joining the vertices. A subgraph of a geometric graph is *non-crossing* if no two edges have a common interior point.

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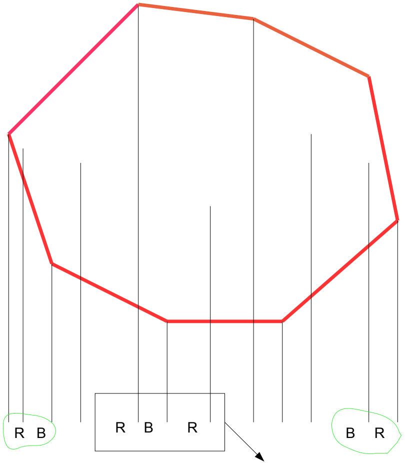
Theorem

In every 2-coloring of a geometric complete graph there is a non-crossing monochromatic spanning tree.

Proof.



There exists an RB – BR consecutive switch – the two red subtrees can be joined by a red edge of the convex hull

Coloring non-complete graphs - the role of independence number

Theorem

(Gy. 1977)If the edges of an arbitrary graph H are colored with two colors, there exists a monochromatic subtree $T \subset H$ with at least $\alpha(H)^{-1}|V(H)|$ vertices.

Coloring non-complete graphs - the role of independence number

Theorem

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Proof. Consider a coloring of the edges of H with two colors. Consider the hypergraph on vertex set $V(H)$ whose edges are the vertex sets of the connected components (in both colors). The dual of this hypergraph is a bipartite graph B. Observe that the maximum number of independent edges in B, $\nu(B)$ satisfies $\nu(B) \leq \alpha(H)$. By König theorem, the edges of B has a transversal of $\nu(B)$ vertices. Some vertex v of this transversal is in at least $\frac{|E(B)|}{\nu(B)} \geq \frac{|V(H)|}{\alpha(H)}$ edges of B. Therefore the component of H corresponding to v has at least $\frac{|V(H)|}{\alpha(H)}$ vertices. □

Problem 1: Largest monochromatic component in 7-colorings

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Question

Let $f(n)$ be the cardinality of the largest monochromatic component that must occur in every 7-coloring of K_n. Then - from previous results - the asymptotic of $f(n)$ is between $\frac{n}{6}$ and $\frac{n}{5}$. Füredi improved the lower bound to $\frac{6n}{35}$. How to improve the upper bound, i.e. 7 colors are really better than 6?

Problem 2: Mono subgraphs with large connectivity

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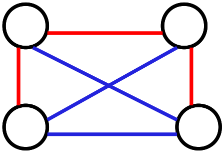
Conjecture

(Bollobás -Gy. 2008.) Every 2-colored K_n contains a monochromatic subgraph that is at least $\frac{n}{4}$-connected.



Problem 3: connectivity plus small diameter

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Problem

(Gy. - Sárközy - Szemerédi, 2008.) Fix k and consider a 2-coloring of K_n. Is it possible to find a color - say red - and a subset $W \subset V(K_n)$ such that $|W| \geq n - ck$ and every pair of W can be connected within W by at least k internally vertex disjoint red paths of length at most three?

Problem 4: largest mono double star

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Question

(Gy. - Sárközy 2008.) For $r \geq 3$*, is there a monochromatic double star of size asymptotic to* $\frac{n}{r-1}$ *in every* r*-coloring of* K_n*? In particular, a mono double star of size* $\frac{n}{2}$ *in every 3-coloring?*

Problem 5: vertex-coverings by components - Lovász - Ryser conjecture

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Conjecture

(Are the following equivalent statements true?

- *In every r-coloring of K_n, $V(K_n)$ can be covered by the vertex sets of at most $r-1$ monochromatic components.*
- *If r partitions are given on a ground set of n elements such that each pair of elements is covered by some block of the partitions then the ground set can be covered by at most $r-1$ blocks.*
- *For every intersecting r-partite (multi)hypergraph $\mathcal{H}$, $\tau(\mathcal{H}) \leq r-1$.*

Conjecture 2 is proved for $r \leq 5$.

Problem 6: vertex-coverings by components in generalized Gallai-colorings

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Question

Gy. 2008. Suppose that $\alpha(G) = 2$ and the edges of G are colored so that no triangle gets three distinct colors. Is it possible to cover the vertices of G by 2008 monochromatic components?

Problem 7: largest component in generalized Gallai-colorings

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Question

Gy. - Sárközy, 2008. Suppose that $\alpha(G) = 2$ *and the edges of* G *are colored so that no triangle gets three distinct colors. How large is the largest monochromatic component? (We know it is between* $\frac{n}{5}$ *and* $\frac{3n}{8}$*.)*

Problem 8: Gallai-colorings of complete hypergraphs

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Question

Gy. - Lehel, 2007. Suppose that K_n^3 (all triples on n vertices) is colored so that no tetrahedron receives four distinct colors. How large is the largest monochromatic component? (We know it is between $\frac{n+3}{2}$ and $\frac{4n}{5}$.)

Thanks for the attention!