The afterlife of a remark of Erdős and Rado

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The leitmotif - a remark of Erdős and Rado

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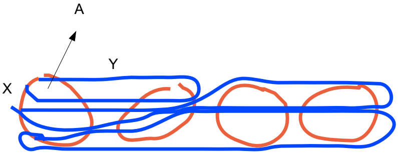
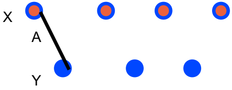
- Pairwise intersecting edges of a bipartite multigraph has a common vertex.
- The previous statements are all equivalent.

The dual of partitions





The dual of two partitions is a bipartite multigraph

Outline of the talk

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- Open problems

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- Coloring by group elements (coloring by group elements and connectivity is zero sum)
- Geometric graphs (points, segments, non-crossing mono subgraphs)

Outline of what is not even mentioned

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- More in "Ramsey theory yesterday, today and tomorrow" - a collection of survey papers to appear in Progress in Mathematics Series, Springer - Birkhäuser.

Type of mono spanning trees in 2-colorings

Height two tree

Octopus

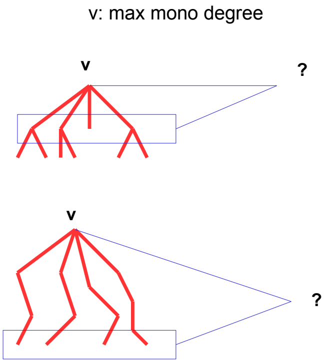
Broom

Height two, octopus

Height two, octopus

Theorem

(Bialostocki, Dierker, Voxman, 1992)In every 2-coloring of K_n there exists a monochromatic spanning octopus and also a monochromatic spanning tree of height at most two.



Broom

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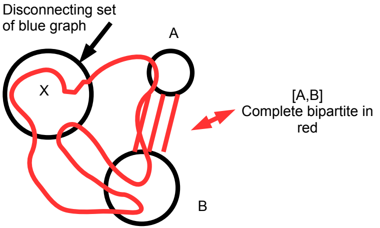
Theorem

(Burr 1992) In every 2-coloring of K_n there exists a monochromatic spanning broom.

w.l.o.g the red graph is at least as connected as the blue graph



Union of a cycle and a complete bipartite graph has a spanning broom!

Spanning diameter three subgraph

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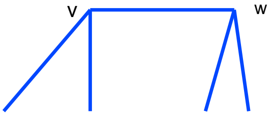
Theorem

(Bialostocki 1992, Mubayi 2002, West 2000) In every 2-coloring of a complete graph there is a monochromatic spanning subgraph of diameter at most three.

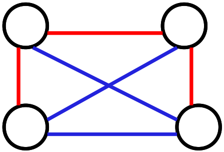
If v,w are at distance > 2 in the red graph then there is a blue spanning double star



Diameter two subgraphs are smaller

Theorem

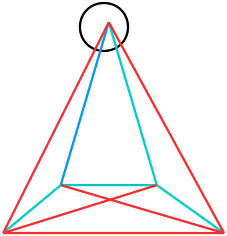
(Erdős - Fowler, 1999) In every 2-coloring of K_n there is a monochromatic subgraph of diameter at most two with at least $\frac{3n}{4}$ vertices. This is sharp as the following figure shows.



2-connected subgraphs

Theorem

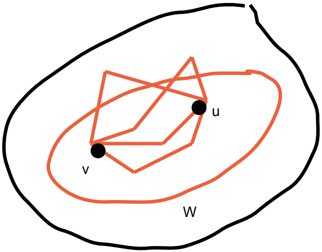
(Bollobás-Gy. 2008) For $n \geq 5$ there is a monochromatic 2-connected subgraph with at least $n-2$ vertices in every 2-coloring of K_n. This is sharp as the following figure shows.



High connectivity plus small diameter

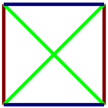
Lemma

(Gy-Sárközy-Szemerédi, 2009) For every k and for every 2-colored K_n there exists $W \subset V(K_n)$ and a color such that $|W| \geq n - 28k$ and any two vertices in W can be connected in that color by k internally vertex disjoint paths, each with length at most three.

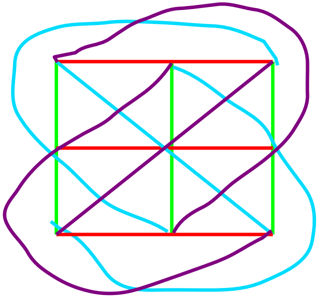


Multicolorings: 3 and 4 colors

Colorings from affine planes

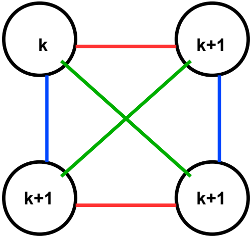


3-coloring



4-coloring

Substitution



Optimal substitution to get 3-coloring of a complete graph on 4k+3 vertices without monochromatic component of 2k+3 vertices

Multicolorings: r colors

Consider an affine plane of order $r-1$ that is r partitions of a ground set $V, |V| = (r-1)^2$ into blocks of size $r-1$ so that each pair of elements of V is covered by a unique block.
For $i = 1, 2, \ldots, r$ color the pairs within the blocks of the i-th partition class with color i.
The size of each monochromatic component is $\frac{1}{r-1}$ fraction of the total number of vertices.

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Largest mono components in r-colorings

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The result can be proved by two different techniques.

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Lemma

(Mubayi 2002 and Liu-Morris-Prince 2004) In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic **double star** with at least $\frac{n}{r}$ vertices.

Large mono component - first proof

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Proof. If there is a red spanning tree we are done. Otherwise the vertex set is spanned by the vertices of an $(r-1)$-colored **complete bipartite graph** which, by the Lemma above, contains a monochromatic double star with at least $\frac{n}{r-1}$ vertices. □

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Corollary

In every r-coloring of K_n there is either a mononochromatic spanning tree or a monochromatic double star with at least $\frac{n}{r-1}$ vertices.

Large mono component - second proof

Assume that the edges of K_n are r-colored. To find a monochromatic component with at least $\frac{n}{r-1}$ vertices is equivalent with finding a vertex of degree at least $\frac{n}{r-1}$ in an intersecting r-partite multihypergraph $\mathcal{H}$ with n edges.

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$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq r-1$$

where D is the maximum degree of $\mathcal{H}$.

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where D is the maximum degree of $\mathcal{H}$.
Thus we have

$$\frac{n}{r-1} = \frac{|E(\mathcal{H})|}{r-1} \leq D(\mathcal{H})$$

.

SOME OPEN PROBLEMS

Question

What can we say about a large monochromatic component? Large means the largest that always there: at least $\frac{n}{r-1}$ vertices. How stable is the extremal coloring - where each component in each color form a complete graph?

Problem 1. Component with a large matching.

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Question

Let $g(n,r)$ be the maximum m such that in every r-coloring of K_n there is a monochromatic component with a matching that covers at least m vertices. Is it true that for any fixed $r \geq 3, g(n,r)$ asymptotic to $\frac{n}{r-1}$? In particular, is $g(n,4)$ $\frac{n}{3}$ true?

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The affirmative answer would imply (through the Regularity lemma) that the r-color Ramsey number of P_n is asymptotic to $(r-1)n$ and would be probably useful in many other applications as well.

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(Ruszinkó, 2010) In every r-coloring of K_n there is a monochromatic subgraph of diameter at most five and with at least $\frac{n}{r-1}$ vertices.

Problem 3. Vertex-coverings by components - Ryser's conjecture

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Conjecture

- *In every r-coloring of a complete graph K, $V(K)$ can be covered by the vertex sets of at most $r-1$ monochromatic components.*
- *For every intersecting r-partite (multi)hypergraph $\mathcal{H}$, $\tau(\mathcal{H}) \leq r-1$. Here τ is the transversal number, the minimum number of vertices that meet all edges.*

Problem 4. When affine plane does not exist - coloring with 7 colors.

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Question

Let $f(n)$ be the cardinality of the largest monochromatic component that must occur in every 7-coloring of K_n. Then - because no affine plane of order 6 exists - the asymptotic of $f(n)$ is between $\frac{n}{6}$ and $\frac{n}{5}$. Füredi improved the lower bound to $\frac{6n}{35}$. How to improve the upper bound, i.e. 7 colors are really better than 6?

Problem 5. Highly connected mono subgraphs in 2-colorings

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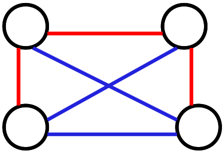
Conjecture

(Bollobás -Gy. 2008.)*Every 2-colored K_n contains a monochromatic subgraph that is at least $\frac{n}{4}$-connected.*



Thanks for the attention!