Vertex covers by monochromatic pieces - results and problems

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Terminology

- coloring: edge coloring (of graphs or hypergraphs). In case of two colors: red and blue
- monochromatic: all edges have the same color
- independence number $\alpha(H)$ of a hypergraph is the cardinality of the largest subset of vertices of H that does not contain any edge of H
- r-color Ramsey number $R(H_1, \dots, H_r)$ is the smallest m for which there is a monochromatic H_i for some i in every r-coloring of the edges of a complete graph (hypergraph) on m vertices. If all H_is are the same H we write $R_r(H)$.
- P_n is the path with n vertices

Subject

To cover (partition) vertices of edge colored graphs (hypergraphs) with vertices of monochromatic

- paths
- cycles
- powers of cycles, bounded degree subgraphs
- connected subgraphs

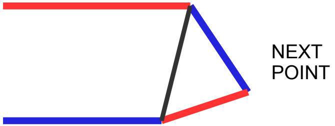
Path partition

Remark

(Gerencsér-Gy., 1967) Vertex set of any 2-colored complete graph K_n can be partitioned into a red and a blue path. The path partition can be found by checking the color of at most $3(n-3)+1$ edges.

Corollary

A 2-colored complete graph has a simple Hamiltonian cycle: a H-cycle which is the union of a red and a blue path.



Cover result implies Ramsey upper bound

Corollary

$R(P_m, P_n) \le m + n - 3$ *if* $m, n \ge 3$.

However, the corollary is not sharp, the Ramsey number is smaller.

Theorem

(Gerencsér-Gy., 1967) $R(P_m, P_n) = m + \lfloor \frac{n}{2} \rfloor - 1$ *for* $m \ge n \ge 2$.

SIMPLE HAMILTONIAN CYCLE

Directed version

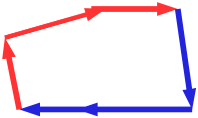
Theorem

(Raynaud, 1973) Complete oriented graph has a simple Hamiltonian cycle in every 2-coloring of its edges.

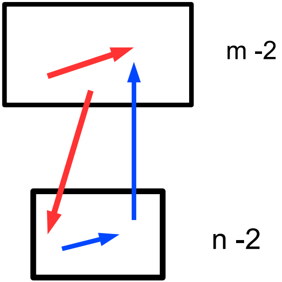
Corollary

If $m, n \geq 3$ *then* $R(\overrightarrow{P_m}, \overrightarrow{P_n}) = m + n - 3.$

SIMPLE DIRECTED HAMILTONIAN
CYCLE



The bound from cover is sharp: $R(\overrightarrow{P_m}, \overrightarrow{P_n}) > m + n - 4.$



Transition from Ramsey number to covering number

Definition

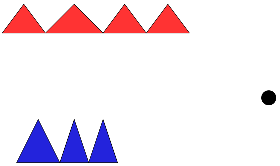
(Gy. - Sárközy - Selkow, 2011) Assume that F is a family of graphs, $1 \le s \le r$ are integers. Let $f(n, s, r, F)$ the largest m such that in every r-coloring of the edges of K_n at least m vertices can be covered by no more than s monochromatic members of F.

- $s = 1$: the Ramsey problem
- cover problem: the smallest s for which $f(n, s, r, F) = n$ - this is the subject of the talk

Hypergraph version - loose (or linear) paths

Conjecture

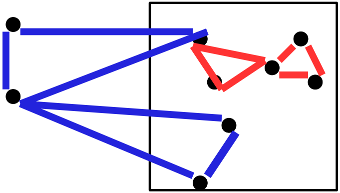
(Gy.- Sárközy, 2013) In any 2-colored complete k-uniform hypergraph there are vertex disjoint red and a blue loose paths covering all but at most $k-2$ vertices. (True with $2k-5$ instead of $k-2$ - thus the case $k=3$ is settled.)



ONE POINT MIGHT BE
UNCOVERED

Example where two disjoint loose paths cannot cover

The box contains odd number of vertices, at least 5.



0	3	RED
1	2	BLUE
2	1	BLUE

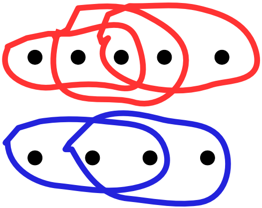
Hypergraph version - tight paths

Question

(Gy.- Sárközy, 2013) Is it possible to partition the vertex set of every 2-colored complete 3-uniform hypergraph into a red and a blue tight path?

Question

Is it possible to partition into 2014 monochromatic tight paths? (It is time to make it easier...)



Extensions to non-complete host graph

Theorem

(Gy. - Jagota - Schelp, 1997) Assume $n \geq 5$ and G is a graph obtained from K_n by deleting at most $m = \lfloor \frac{n}{2} \rfloor$ edges. Then for every 2-coloring of G, $V(G)$ can be partitioned into a red and a blue path. (Best possible, not true if m is replaced by $m+1$).

Theorem

(Schaudt - Stein, 2014) Let G be a 2-colored complete t-partite graph, $t \geq 3$, with no partite class of size larger than $\frac{|V(G)|}{2}$. Then $V(G)$ can be partitioned into a red and a blue path.

Cycles instead of paths - Lehel's conjecture

Lehel conjectured (1979) that the vertex set of any 2-colored complete graph can be partitioned into a red and a blue cycle. Empty set, one vertex and an edge is accepted as a cycle.

Theorem

(Gy., 1983) Vertex set of any 2-colored complete graph can be covered by a red and a blue cycle that intersect in at most one vertex.

Theorem

(Luczak - Rödl - Szemerédi, 1998, Allen, 2008) Lehel's conjecture is true for large enough complete graphs.

Theorem

(Bessy - Thomassé 2010) Lehel's conjecture is true.

Conlon and Stein (2014) extended Bessy - Thomassé theorem (using it as a black box) to locally 2-colored complete graphs.

Infinte version - extends to multicoloring!

Theorem

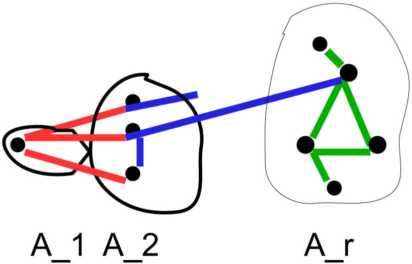
(Rado, 1987) Countable infinite r-colored complete graphs can be partitioned into finite or one-way infinite monochromatic paths of DIFFERENT colors. Consequently, at most r monochromatic paths cover the vertex set.

$r-1$ monochromatic paths cannot always cover

Edge xy gets color

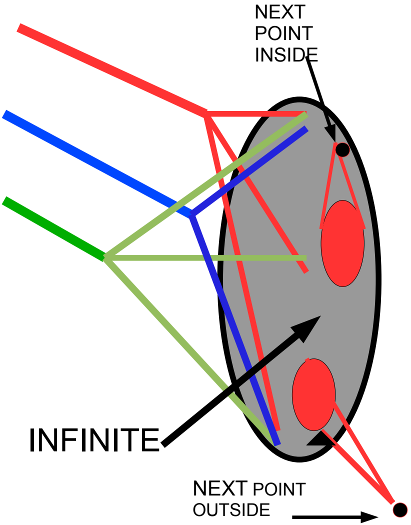
$$\min\{i : A_i \cup \{x,y\} \neq \emptyset\}$$

where $|A_i| = 2^i - 1$ for $i = 1, 2, \ldots, r-1$ and $|A_r| = \infty$ ($\geq 2^r - 1$ in finite case).



Standard coloring

Rado's proof



A misunderstanding - when red and blue cannot mix

I mentioned once to Paul Erdős that for any 2-coloring of K_n, $pathc(K_n) \leq 2$ and he said he did not believe it. It soon turned out that Paul thought that the covering paths must be monochromatic IN THE SAME color... Indeed, with this definition the path cover number depends on n.

Theorem

(Erdős - Gy., 1995) The vertex set of any 2-colored K_n can be covered by at most $2\sqrt{n}$ monochromatic paths of the same color.

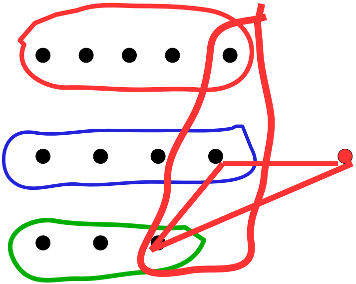
Conjecture

(Erdős - Gy., 1995) The vertex set of any 2-colored K_n can be covered by at most $\sqrt{n}$ monochromatic paths of the same color. This would be best possible.

Hypergraph version - Berge paths behave well!

Theorem

(Gy.- Sárközy, 2013) Vertex set of any k-uniform r-colored complete hypergraph K can be partitined into monochromatic Berge paths of DIFFERENT colors. Consequently, at most r Berge paths cover $V(K)$.



Infinite hypergraph version?

The proof of Rado can be easily extended to loose paths.

Theorem

(Gy.- Sárközy, 2013) Vertex set of any countably infinite k-uniform r-colored hypergraph can be partitined into monochromatic loose paths of DIFFERENT colors.

It is more tricky to extend it further to tight paths.

Theorem

(M. Elekes - D. Soukup - L. Soukup - Z. Szentmiklóssy, 2014) Vertex set of any countably infinite k-uniform r-colored hypergraph can be partitined into monochromatic tight paths of different colors.

Notation for path and cycle cover (partition)

Definition

The path (cycle) partition number of an colored graph G is the minimum number of vertex disjoint monochromatic paths (cycles) whose vertices cover the vertices of G. These numbers are denoted by $pathp(G), cyclep(G)$, respectively. Similarly, $pathc(G), cyclec(G)$ denote the minimum number of monochromatic paths (cycles) whose vertices cover the vertices of G.

Complete graph multicolor path and cycle cover conjectures

Conjecture

(Gy., 1989) For any r-colored complete graph K, $pathp(K) \leq r$. Weaker version: $pathc(K) \leq r$.

Conjecture

(Erdős - Gy. - Pyber, 1991) For any r-colored complete graph K, $cyclep(K) \leq r$.

Advances on path cover and partition problem

It is not obvious that $pathc(K)$ depends on r only - not on $|V(K)|$.

Theorem

(Gy., 1989) For any r-colored complete graph K, $pathc(K) \leq r^4 + r^2 + 1$.

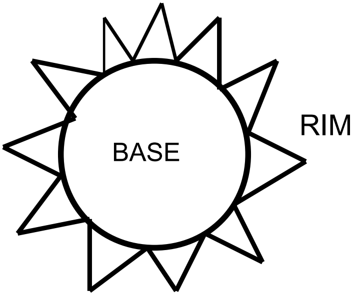
The next result extends this from cover to partition, from paths to cycles - and the bound is better...

Theorem

(Erdős - Gy. - Pyber, 1991) For any r-colored complete graph K, $cyclep(K) \leq cr^2 \log r$.

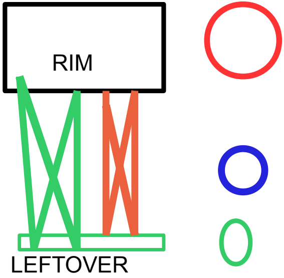
Partition into monochromatic cycles - step 1

1. Find an absorbing monochromatic starter of size cn in r-colored K_n - a base cycle with a rim



Partition into monochromatic cycles - steps 2,3

2.Find repeatedly disjoint monochromatic cycles until only a small set of leftover vertices remain (relative to the size of the rim of the starter).
3. Cover the leftover from the rim - the part that remains from the starter is still a monochromatic cycle. Here Pósa's cycle cover lemma is used: the vertex set of any graph G can be covered by at most $\alpha(G)$ cycles, edges and vertices.



Improving the bound $cyclep(K) \leq cr^2 \log r$

Theorem

(Gy. - Ruszinkó - G.Sárközy - Szemerédi, 2006) For any r-colored complete graph K_n with $n \geq n(r)$, $cyclep(K_n) \leq 100r \log r$.

The gain of a factor r comes from two refinement of the outlined method. First the Regularity Lemma is used to replace the starter with a connected structure of dense monochromatic regular pairs (lifting a dense connected monochromatic matching of the cluster graph). The second improvement is a more efficient covering of the leftover vertices. In fact, the essence of a nearly optimal cover strategy is in the following lemma.

Lemma

(Gy. - Ruszinkó - G.Sárközy - Szemerédi, 2006) Assume $[A, B]$ is an r-colored complete bipartite graph, $|A| \geq r|B|$. Then B can be covered by at most r vertex disjoint monochromatic CONNECTED MATCHINGS.

3-color case of path cover and partition problem

Theorem

(Gy. - Ruszinkó - G.Sárközy - Szemerédi, 2011) Apart from $o(n)$ vertices, the vertex set of any 3-colored K_n can be partitioned into 3 monochromatic cycles.

Main idea of proof: For even n, the vertices of any 3-colored K_n can be partitioned into three monochromatic CONNECTED matchings.

Theorem

(Pokrovskiy, 2012) For any 3-colored complete graph K, $pathp(K) \leq 3$.

Main idea of proof: Any 2-colored complete graph K can be partitioned into a red path and a blue BALANCED complete bipartite graph.

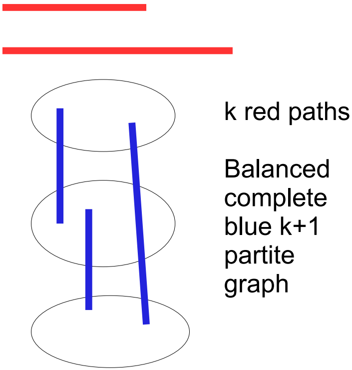
Theorem

(Pokrovskiy, 2012) Vertex set of any 3-colored complete graph can be covered by 3 monochromatic paths of DIFFERENT colors. (Not true for partitions.)

Pokrovskiy lemma

Lemma

For any k, the vertex set of any 2-colored complete graph can be partitioned into k vertex disjoint red paths and a disjoint blue complete balanced $(k+1)$-partite graph.



An application of Pokrovskiy lemma

Theorem

(Pokrovskiy, 2013) $R(P_n, P_n^k) = (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$ *where* P_n^k *is the* k*-th power of* P_n.

Paths or cycles - does it make a difference?

For 2-colored complete graph K

- $pathp(K) \leq 2$ is immediate
- $cyclec(K) \leq 2$ is easy (Gy., 1983)
- $cyclep(K) \leq 2$ - Lehel's conjecture - was difficult!

For 3-colored complete graph K

- $pathp(K) \leq 3$ is true
- $cyclep(K) \leq 3$ (Gy. - Erdős -Pyber conjecture) is not (quite) true, one vertex left uncovered... (Pokrovskiy 2014).

Conjecture

(Pokrovskiy, 2012) In every r-coloring of a complete graph K there are r vertex disjoint monochromatic cycles covering all but C_r vertices of K.

Balanced complete bipartite host graph

Theorem

(Gy., 1989) For r-colored $K_{n,n}$, $pathc(K_{n,n})$ is bounded by a function of r.

Theorem

(Haxell, 1997) For r-colored $K_{n,n}$, $cyclep(K_{n,n})$ is bounded by a function of r, for large r, $cyclep(K_{n,n}) \leq c(r\log(r))^2$.

Conjecture

(Pokrovskiy, 2014) For r-colored $K_{n,n}$, $pathp(K_{n,n}) \leq 2r-1$ (if true best possible). True for $r=2$.

Theorem

(D.Soukup, 2014) For r-colored balanced countably infinite complete bipartite graph B, $pathp(B) \leq 2n-1$.

Hypergraphs - loose and tight cycles

The method of Erdős - Gy. - Pyber for bounding the cycle partition number can be extended to loose cycles of hypergraphs.

Theorem

(Gy.- Sárközy, 2013) There exists $c(r,k)$ such that the vertex set of any r-colored complete k-uniform hypergraph can be covered by at most $c(r,k)$ vertex disjoint monochromatic loose cycles.

Bound on $c(r,k)$ is obtained from the strong hypergraph Regularity Lemma of Rödl and Schacht, with the hypergraph Blow-up lemma of Keevash.

Theorem

(Sárközy, 2014) $c(r,k) \le 50rk \log rk$.

Problem

(Sárközy, 2014) Prove that there exists $c(r,k)$ bounding the tight cycle partition number of r-colored complete k-uniform hypergraphs.

Pósa's cycle cover theorem and possible extensions

Theorem

(Pósa, 1963) The vertex set of any graph G can be covered by at most $\alpha(G)$ cycles, edges and vertices.

How to extend it to hypergraphs?

Conjecture

(Gy. - Sárközy, 2014) Every k-uniform hypergraph can be covered by at most $\alpha(H)$ loose cycles and parts of hyperedges. (The case $k = 2$ is Pósa's theorem.)

The conjecture is true if loose cycles replaced by weak cycles (sequence of edges in which only the cyclically consecutive edges intersect) or by loose paths.

Graphs with fixed independence number

Theorem

(Sárközy, 2010) The vertex set of any r-colored graph G with $\alpha = \alpha(G)$, $V(G)$ can be partitioned into at most $25(\alpha r)^2 \log \alpha r$ monochromatic cycles.

Conjecture

(Sárközy, 2010) For any r-colored graph G, cyclep$(G) \leq \alpha(G)r$.

Pokrovskiy's example makes the conjecture false but it is probably true in a somewhat weaker form. The 2-color case is true in asymptotic sense.

Theorem

(Balogh - Barát - Gerbner - Gy. - Sárközy, 2012) For every $\eta > 0$ and for every positive integer α, there is $n_0(\eta, \alpha)$ such that the following holds. For every n-vertex graph G with $n \geq n_0$ and $\alpha(G) = \alpha$, in every 2-coloring of G there exists at most 2α vertex disjoint monochromatic cycles, covering all but at most ηn vertices of G.

Hypergraphs with fixed independence number

Theorem

(Gy. - Sárközy, 2014) For every r-coloring of a k-uniform hypergraph H with $\alpha = \alpha(H)$, $V(H)$ can be partitioned into at most $c(r, k, \alpha)$ monochromatic loose cycles.

Conjecture

(Gy. - Sárközy, 2014) Every k-uniform hypergraph can be covered by at most $\alpha(H)$ loose cycles and parts of hyperedges. (The case $k = 2$ is Pósa's theorem.)

The conjecture is true if loose cycles replaced by weak cycles (sequence of edges in which only the cyclically consecutive edges intersect) or by loose paths.

Host graphs with large minimum degree

Theorem

(Balogh - Barát - Gerbner - Gy. - Sárközy, 2012) For every $\eta > 0$ there is $n_0(\eta)$ such that the following holds. For every n-vertex graph G with $n \geq n_0$ and $\delta(G) > (\frac{3}{4} + \eta)n$, in every 2-coloring of G there exists a red and a blue cycle, covering all but at most ηn vertices of G.

Theorem

(Barát - Sárközy, 2014) For every $\eta > 0$ there is $n_0(\eta)$ such that the following holds. For every n-vertex graph G with $n \geq n_0$ and such that for any two non-adjacent vertices x, y, $deg(x) + deg(y) \geq (\frac{3}{2} + \eta)n$, in every 2-coloring of G there exists a red and a blue cycle, covering all but at most ηn vertices of G.

Problem

Find stronger forms of the results above.

Covers with graphs of bounded degree - I.

Theorem

(Sárközy - Selkow, 2000) Every r-colored complete graph can be covered by at most $r^{c(r \log r + d)}$ vertex disjoint connected monochromatic d-regular graphs and vertices (c is an absolute constant).

Theorem

(Sárközy, 2013) There exists $c(k)$ such that the vertex set of every 2-colored complete graph can be covered by at most $c(k)$ vertex disjoint k-th power of a cycle.

Covers with graphs of bounded degree - II.

A sequence of graphs $F = \{F_1, F_2, \dots\}$ is Δ-bounded if each F_i has i vertices and maximum degree at most Δ.

Theorem

(Grinshpun - Sárközy, 2014) There is an absolute constant C such that for every Δ and every Δ-bounded sequence F, any 2-colored complete graph can be partitioned into at most $2^{C\Delta \log \Delta}$ vertex disjoint monochromatic members of F. If all F_is are bipartite graphs, $2^{C\Delta}$ members suffice and this is best possible apart from the constant in the exponent.

Covers with connected pieces - Ryser's conjecture

A subset S of vertices in a colored hypergraph is called CONNECTED in color i if the edges of color i within S form a connected subhypergraph. S is CONNECTED if it is connected in some color.

Definition

The connected cover (partition) number of an colored graph or hypergraph H is the minimum number of (vertex disjoint) connected subsets covering $V(H)$. These numbers are denoted by $conc(H), conp(H)$, respectively.

A conjecture of Ryser (1971) can be formulated as follows.

Conjecture

For every r-colored graph G, $conc(G) \leq (r-1)\alpha(G)$ $(r \geq 2)$.

Ryser's conjecture is known to be true for $\alpha(G) = 1, r \leq 5$; $r = 2$ (equivalent with König's theorem, $r = 3$ (Aharoni, 2001). Ryser's conjecture is sharp if $r - 1$ is a prime power

Covers with connected pieces - complete bipartite host

Conjecture

(Gy. - Lehel, 1977) For every r-colored complete bipartite graph G, $conc(G) \leq 2r - 2$.

Theorem

(G.Chen - Fujita - Gy. - Lehel - Á. Tóth, 2013) The conjecture is true for $r \leq 5$.

Remark

The conjecture can be easily reduced to r-colorings where each color class is the union of vertex disjoint complete bipartite graphs, covering all vertices. Such a reduction is not known for Ryser's conjecture.

Covers with connected pieces - hypergraphs

Although Ryser's conjecture is open for complete graphs ($\alpha(G) = 1, r \geq 6$), it was a surprise that its analogue for complete k-uniform hypergraphs comes easily.

Theorem

(Z. Király, 2010) If $k \geq 3$, for every r-colored complete k-uniform hypergraph H, $conc(H) \leq \lceil \frac{r}{k} \rceil$. The bound is sharp for $k \geq 3, r \geq 1$.

For non- complete k-uniform hypergraphs H ($\alpha(H) \geq k$) only the initial steps are known and there is no plausible conjecture.

Theorem

(Fujita - Furuya -Gy. - Ágnes Tóth, 2014) For $k \geq 2$ every kcolored k-uniform hypergraph H with $\alpha(H) = k$ satisfies $conc(H) \leq 2$. For $k \geq 3$ every $(k+1)$-colored k-uniform hypergraph H with $\alpha(H) = k$ satisfies $conc(H) \leq 3$. Both results are sharp.

Partitions into connected pieces

Conjecture

(Erdős - Gy. - Pyber, 1991) For every r-colored complete graph K, $conp(K) \leq r-1$. (True for $r \leq 3$.)

Theorem

(Haxell - Kohayakawa, 1996) If n is large compared to r then for every r-colored complete graph K, $conp(K) \leq r$.

Theorem

(Fujita - Furuya -Gy. - Ágnes Tóth, 2012) For every 2-colored non-trivial k-uniform hypergraph H, $conp(H) \leq \alpha(H) - r + 2$.

Corollary

(F.- F. -Gy. - Á. T., 2012) For every 2-colored non-trivial k-uniform hypergraph H, $conp(H) \leq \alpha(H)$. This is an extension of König's theorem.

Thanks for the attention!