How to generalize Gallai colorings?

András Gyárfás
Computer and Automation Research Institute
Hungarian Academy of Sciences
Budapest, P.O. Box 63
Budapest, Hungary, H-1518

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Outline of the talk

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- Definition of Gallai colorings (G-colorings)

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- A fast overview of basic results
- Possible generalizations - initial results and open problems

Gallai-colorings

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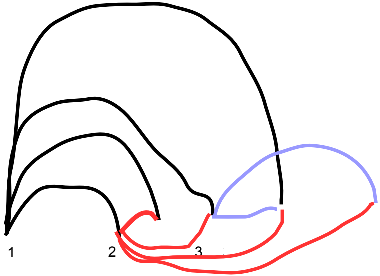
- 2-colorings
- canonical colorings



CANONICAL G-COLORING

Substitution property of G-colorings

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Theorem

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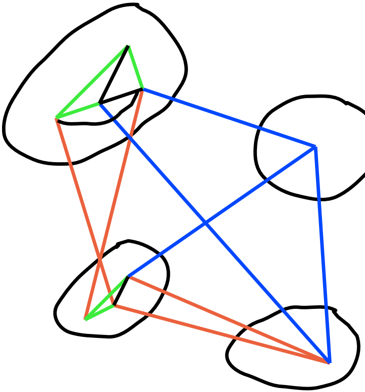
This formulation is from Gy. - Simonyi 2004, but already implicit in Gallai 1967 and more general decomposition theorems exist, Cameron - Edmonds 1997, ...

Substitution



Substituting into red-blue coloring

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- Disconnection property implies substitution property easily

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All - except 3. - can be obtained as 'black-box' extensions from the corresponding 2-coloring results...(Gy-Sárközy-Sebő-Selkow to appear)

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- 3. Hypergraphs - forbidding rainbow tetrahedron?
- 4. Directed graphs: forbidding rainbow cyclic (transitive) triangle?

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- If H has disconnection property then it is close to bipartite graphs: becomes bipartite by deleting at most two edges.
- A unicyclic graph H has the disconnection property if its cycle is a triangle.
- A research project (Fujita, Gy., Magnant, Martin, Ruszinkó, Sárközy, Selkow, Seress) with some initial results.

G-coloring of non-complete graphs

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Definition

G-coloring of a graph is a coloring of the edges of G so that no triangle receives three distinct colors.

- Do we have connection property?
- What happens for two colors?

Coloring non-complete graphs - the role of independence number

Theorem

(Gy. 1977)If the edges of an arbitrary graph H are colored with two colors, there exists a monochromatic subtree $T \subset H$ with at least $\alpha(H)^{-1}|V(H)|$ vertices.

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Proof. Consider a coloring of the edges of H with two colors. Consider the hypergraph on vertex set $V(H)$ whose edges are the vertex sets of the connected components (in both colors). The dual of this hypergraph is a bipartite graph B. Observe that the maximum number of independent edges in B, $\nu(B)$ satisfies $\nu(B) \leq \alpha(H)$. By König theorem, the edges of B has a transversal of $\nu(B)$ vertices. Some vertex v of this transversal is in at least $\frac{|E(B)|}{\nu(B)} \geq \frac{|V(H)|}{\alpha(H)}$ edges of B. Therefore the component of H corresponding to v has at least $\frac{|V(H)|}{\alpha(H)}$ vertices. □

Connection property

Theorem

(Gy.- G. Sárközy, 2008) If the edges of an arbitrary graph H are G-colored, there exists a monochromatic subtree with at least

$$(\alpha^2(H) + \alpha(H) - 1)^{-1}|V(H)|$$

vertices.
On the other hand, there are G-colored graphs H^* with largest monochromatic connected subgraph order of magnitude

$$\frac{log \alpha(H^*)}{\alpha^2(H^*)}|V(H^*)|.$$

Question

Gy. - Sárközy, 2008. Suppose that $\alpha(G) = 2$ and we have a G-coloring on G. How large is the largest monochromatic component? (We know it is between $\frac{n}{5}$ and $\frac{3n}{8}$.)

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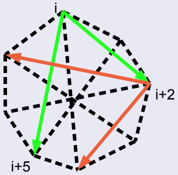

G-coloring of the the complement of the Wagner graph by 8 colors

Connection property for hypergraphs

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Theorem

(Gy. - G. Sárközy, 2008) Suppose that the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that $\mathcal{H}$ does not contain rainbow copies of an r-uniform hypergraph F. Then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \geq c|V(\mathcal{H})|$ where c depends only on $F, r, \alpha(\mathcal{H})$.

Covering property?

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Question

Gy. 2008. Suppose that $\alpha(G) = 2$ and a G-coloring is given on G. Is it possible to cover the vertices of G by 2008 (2009) monochromatic components?

Gallai-colorings of hypergraphs

- Define G-colorings of K_n^3 (all triples on n vertices) as colorings of the triples so that no tetrahedron receives four distinct colors.

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Theorem

Gy. - Lehel, 2007. Let f(n) be the size of the largest monochromatic component one can find in any G-coloring of K_n^3*. Then* $\frac{n+3}{2} \leq f(n) \leq \frac{4n}{5}$.

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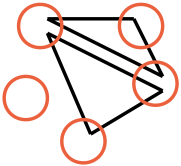
- Upper bound:



G-coloring of the triples with 5 colors

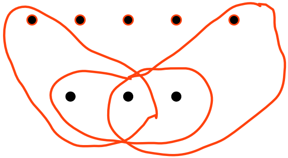
lower bound

lower bound

Assign two triples of the same color to any quadruple of the vertex set. Some triple, say red, is in at least

$$\frac{2\binom{n}{4}}{\binom{n}{3}} = \frac{n-3}{2}$$

quadruples, giving a red component of size at least $\frac{n+3}{2}$.



Digraphs

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