# Erdős for Beginners

The Mathematics of Paul Erdős for Non-mathematicians

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# An early visit to Memphis

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- The individual record is held by A. Sárközy with 62 joint papers... (Faudree, Schelp, Rousseau hold 3-5 places with 50,42,35)

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- More than 100 problems in the American Mathematical Monthly...

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- A testimony?
- "A well-chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. It might be a 'marshmallow' serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be an 'acorn', requiring deep and subtle new insights from which a mighty oak can develop."
- From Paul Erdős: *Some of my favorite problems and results - 1997*

# Just from the oven...

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- "This prompted me to conjecture: is it true that a set of $n^2$ points in the plane always contains $2n-2$ points which do not determine a right angle? - Perhaps this is too optimistic - if a reader finds a counterexample, 2n-2 should be replaced by $cn$. I can only prove $cn^{2/3}$ instead of $cn$." - 1977

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- "Let me close this section with a few problems about (my old friends) the primes. In fact, my very first paper was on a new proof of Chebysev's theorem:
- Chebysev said it and I'll say it again:
  there is always a prime between $n$ and $2n$"

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- "I conjecture (6). The proof of (6) probably will not be very difficult, but I did not do it as yet."

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- "I offered 1000 dollars...At the meeting held last year in Keszthely Kostochka proved ... I immediately handed him a consolation prize of 100 dollars." - 1994

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- "Anyone who has had the privilege of attending a talk of Paul Erdős will certainly enjoy both the tone and content of this paper. Those without that exposure will experience both the flavor of his wide expertise and his outstanding technique of describing problems." - D.W.Vanderjagt, Math. Reviews 1998

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- **10282 [1993]. Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest, Hungary.**
- **114 problems altogether!**

# Erdős with Memphis students

# Solution to his child prodigy test

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- **Solution by Martha Wachsberger and E. Weiszfeld, Budapest, Hungary.** We may write

$$a_r = 2^{b_r} c$$

where $c$ is an odd integer. Since the number of odd integers less than $2n$ is n, there are at least two $a$-s e.g. $a_i$ and $a_k$ having the same $c$; and it is then evident that one of these is a divisor of the other.

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- In the spirit of Erdős, I offer two dollars for the solution [*offer limited to students and non-mathematicians*]

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- *from "Some problems I presented or planned to present in my short talk" 1995*

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- Erdős refers to consequences due to the discovery of the undecidability of the Continuum Hypothesis.
- And consequently some old problems of Erdős about infinite sets became undecidable - true or false whatever you like better...

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- *Is NATURAL INN larger than EVEN INN?*

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- *Which hotel is larger?  They are EQUAL...*
- NATURAL INN: the rooms are numbered as $\{1, 2, 3, 4, 5, 6 \ldots ...\}$
- *Is NATURAL INN larger than EVEN INN?*
- They are EQUAL because of the EXCHANGE PLAN (one-to-one correspondence):
  $1 \rightleftarrows 2,\ 2 \rightleftarrows 4,\ 3 \rightleftarrows 6,\ 4 \rightleftarrows 8,\ 5 \rightleftarrows 10, \ldots$

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- NATURAL INN < ??? < REAL INN
- UNDECIDABLE! (Proved by Cohen 1962, relying also on Gödel 1931).

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- "Over sixty years ago Turán and I thought that any integer sequence with positive density must contain arbitrary long arithmetic progressions."
- A sequence of density $.02$ must contain 2 percent of the numbers $1, 2, 3, \ldots, n$ for all $n$.

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- Both approaches - with some new directions - brought spectacular results in the last decade.

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$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty$$
- "I offer 5000 dollars for a proof (or disproof) of this. Neither Szemerédi nor Fürstenberg's methods are able to settle this but perhaps the next century will see its resolution."

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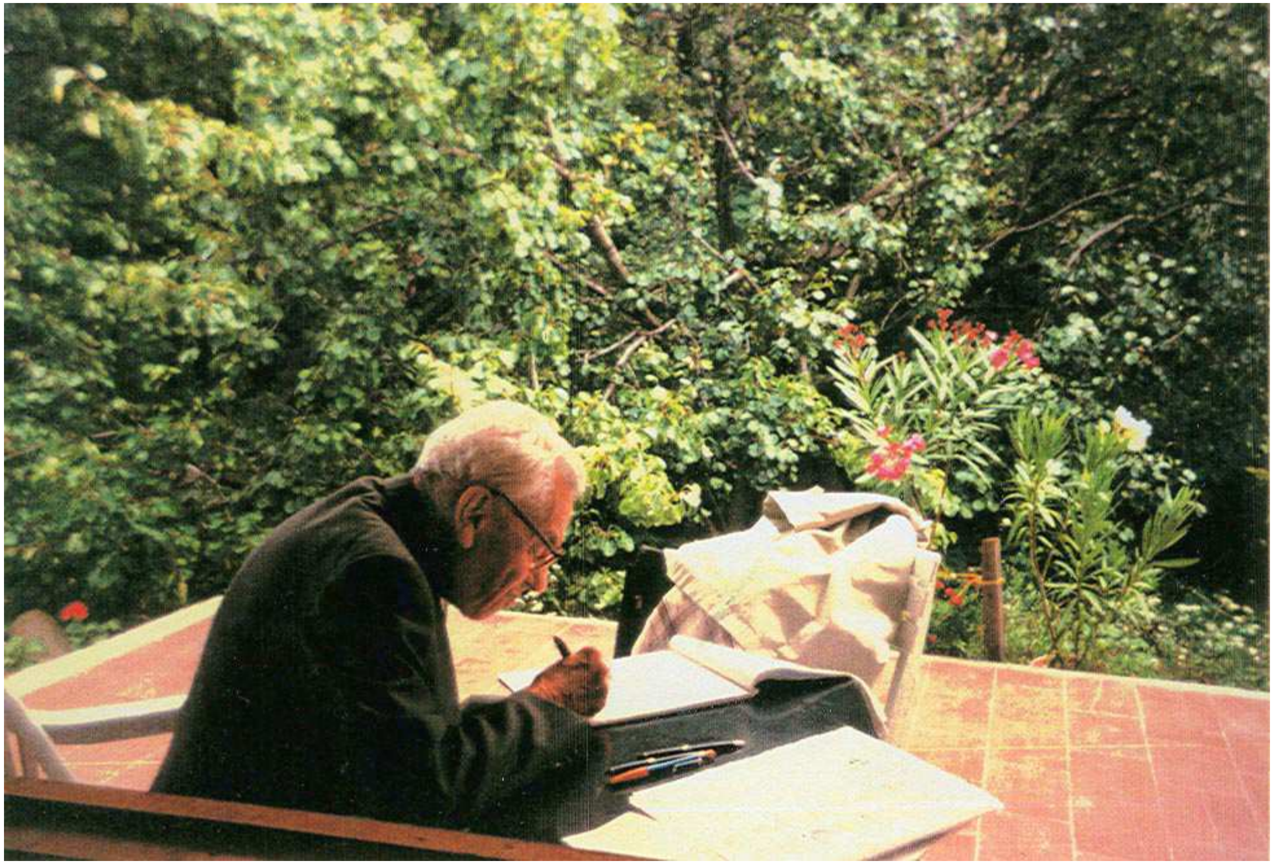
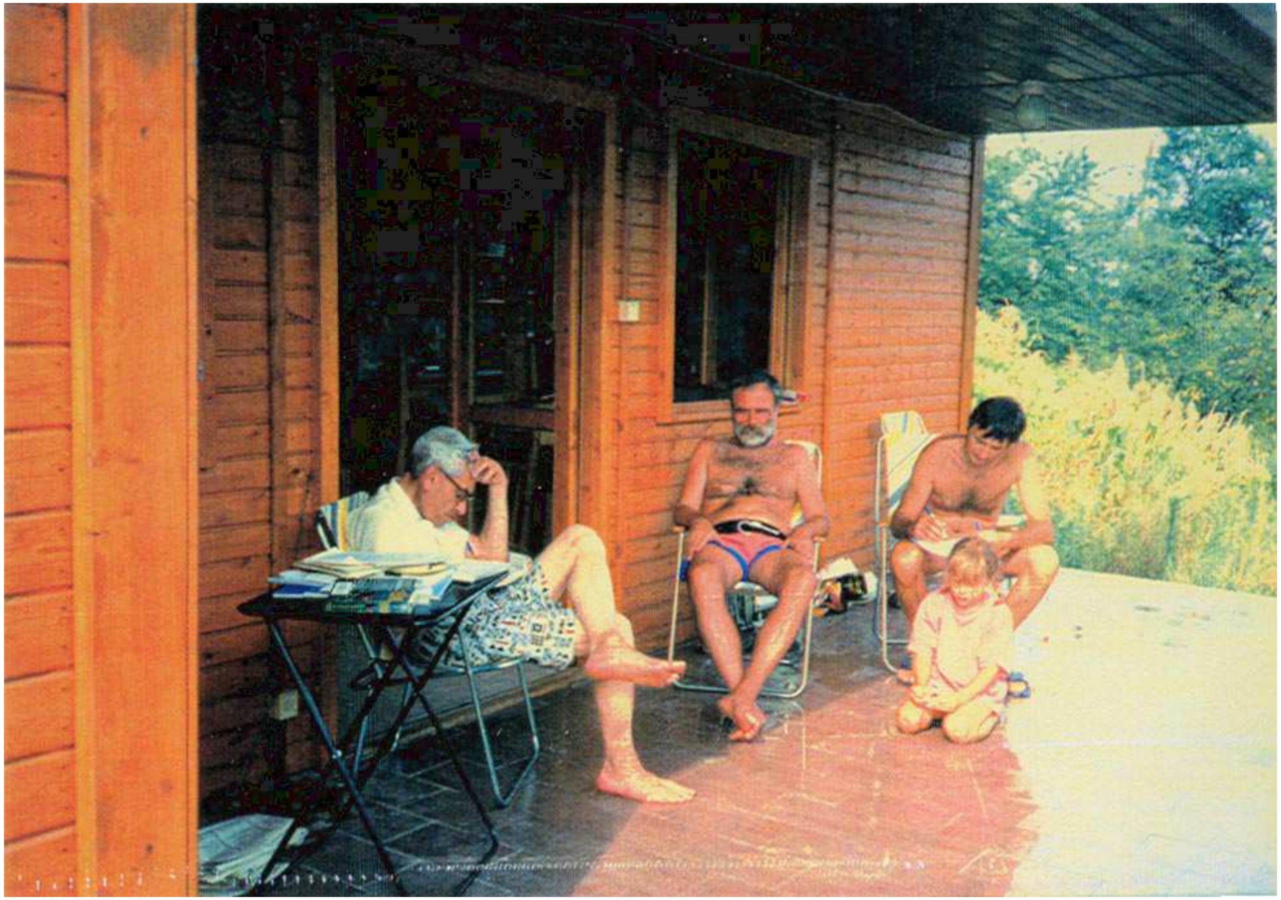
- Erdős - in the years 1993 - 1996 - liked to spend some weeks with us there as a family guest.

# Erdős works on a problem paper



# Working on an Erdős problem



# Going to dinner

# Reading fiction - no mathematics!

# The last summer - 1996

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- The debate about the upper and lower bound is not resolved ever since... We submitted our paper on September 15, 1996. It turned out that only four more days left for Paul to create problems, conjectures and proofs.
- Nevertheless, the acorns he planted are around us and many of them are still growing to mighty oaks.