Partition of graphs and hypergraphs into monochromatic connected parts

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Abstract

We show that two results on covering of edge colored graphs by monochromatic connected parts can be extended to partitioning. We prove that for any 2-edge-colored non-trivial \( r \)-uniform hypergraph \( H \), the vertex set can be partitioned into at most \( \alpha(H) - r + 2 \) monochromatic connected parts, where \( \alpha(H) \) is the maximum number of vertices that does not contain any edge. In particular, any 2-edge-colored graph \( G \) can be partitioned into \( \alpha(G) \) monochromatic connected parts, where \( \alpha(G) \) denotes the independence number of \( G \). This extends König’s theorem, a special case of Ryser’s conjecture.

Our second result is about Gallai-colorings, i.e. edge-colorings of graphs without 3-edge-colored triangles. We show that for any Gallai-coloring of a graph \( G \), the vertex set of \( G \) can be partitioned into monochromatic connected parts, where the number of parts depends only on \( \alpha(G) \). This extends its cover-version proved earlier by Simonyi and two of the authors.

1 Introduction

In this paper we prove two results about partitioning edge-colored graphs (and hypergraphs) into monochromatic connected parts. Let \( k \) be a positive integer. A \( k \)-edge-colored (hyper)graph is a (hyper)graph whose edges are colored with \( k \) colors. It was observed in [5] that a well-known conjecture of Ryser which was stated in the thesis of his student Henderson [11] can be formulated as follows.

\[ \alpha(G) \leq \frac{|\text{edges}|}{\text{colors}} \]

This theorem is a special case of Ryser’s conjecture. Our second result is about Gallai-colorings, i.e. edge-colorings of graphs without 3-edge-colored triangles. We show that for any Gallai-coloring of a graph \( G \), the vertex set of \( G \) can be partitioned into monochromatic connected parts, where the number of parts depends only on \( \alpha(G) \). This extends its cover-version proved earlier by Simonyi and two of the authors.
**Conjecture 1.** If the edges of a graph are colored with \( k \) colors then \( V(G) \) can be covered by the vertices of at most \( \alpha(G)(k - 1) \) monochromatic connected components (trees).

Ryser’s conjecture (thus Conjecture 1) is known to be true for \( k = 2 \) (when it is equivalent to König’s theorem). After partial results [9], [13], the case \( k = 3 \) was solved by Aharoni [1], relying on an interesting topological method established in [2]. Recently Király [12] showed, somewhat surprisingly, that an analogue of Conjecture 1 holds for hypergraphs: for \( r \geq 3 \), in every \( k \)-coloring of the edges of a complete \( r \)-uniform hypergraph, the vertex set can be covered by at most \( \lfloor \frac{k}{r} \rfloor \) monochromatic connected components (and this is best possible). The authors in [4] will consider extensions of Király’s result for non-complete hypergraphs.

The strengthening of Conjecture 1 from covering to partition was suggested in [3] (and proved for \( k = 3, \alpha(G) = 1 \)). In this paper we extend the \( k = 2 \) case of Conjecture 1 for hypergraphs and for partitions instead of covers (Theorem 4).

Our second partition result (Theorem 6) is about Gallai-colorings of graphs where the number of colors is not restricted but 3-edge-colored triangles are forbidden. This extends the main result of [8] from cover to partition.

We consider hypergraphs \( H \) with edges of size at least two, i.e. we do not allow singleton edges. Let \( V(H), E(H) \) denote the set of vertices and the set of edges of \( H \), respectively. A hypergraph is \( r \)-uniform if all edges have \( r \geq 2 \) vertices (graphs are 2-uniform hypergraphs). When there is no fear of confusion in context, we just say hypergraphs briefly. A hypergraph \( H \) without any edge is called trivial. The cover graph \( G_H \) of a hypergraph \( H \) is the graph defined by the pairs of vertices covered by some hyperedge; namely, \( G_H \) is the graph on \( V(H) \) such that \( e \in E(G_H) \) if and only if \( e \) is covered by some hyperedge of \( H \).

The definition of independence number of hypergraphs is not completely standard. The independence number \( \alpha(H) \) is the cardinality of a largest subset \( S \) of \( V(H) \) that does not contain any edge of \( H \) (i.e., the maximum number of vertices in an induced trivial subhypergraph of \( H \)). Another useful variant important in this paper is the strong independence number \( \alpha_1(H) \), the cardinality of a largest subset \( S \) of vertices such that any edge of \( H \) intersects \( S \) in at most one vertex. In fact, \( \alpha_1(H) = \alpha(G_H) \). For example, if \( H \) is the Fano plane, \( \alpha_1(H) = 1, \alpha(H) = 4 \). For a complete \( r \)-uniform hypergraph \( H, \alpha_1(H) = 1, \alpha(H) = r - 1 \). For \( r \)-uniform hypergraphs these numbers are linked by the following inequality.

**Proposition 1.** For any non-trivial \( r \)-uniform hypergraph \( H \), we have \( \alpha_1(H) \leq \alpha(H) - r + 2 \).  

**Proof.** Suppose that \( S \) is strongly independent in \( H \). Take any \( e \in E(H) \) (it satisfies \( |S \cap e| \leq 1 \) by the definition of \( S \)) and any \( v \in e \setminus S \). Then the set \( T = (S \cup e) \setminus \{v\} \) is independent and \( |T| \geq |S| + r - 2 \). \( \square \)

We need the simplest extension of connectivity from graphs to hypergraphs (no topology involved). A hyperwalk in \( H \) is a sequence \( v_1, e_1, v_2, e_2, \ldots, v_{t-1}, e_{t-1}, v_t \), where for all \( 1 \leq i < t \)
we have \( v_i \in e_i \) and \( v_{i+1} \in e_i \). We say that \( v \sim w \), if there is a hyperwalk from \( v \) to \( w \). The relation \( \sim \) is an equivalence relation, and the subhypergraphs induced by its classes are called the connected components of the hypergraph \( H \). A vertex \( v \) that is not covered by any edge forms a trivial component with one vertex \( v \) and no edge. The vertex sets of the connected components of a hypergraph \( H \) coincide with the vertex sets of the connected components of \( G_H \).

Let \( H \) be an edge-colored hypergraph. For a subset \( S \) of \( V(H) \), the subhypergraph induced by \( S \) in \( H \), that is the hypergraph on the vertex set \( S \) with edge set \( \{ E \in E(H) \mid E \subseteq S \} \), is denoted by \( H[S] \). A vertex partition \( \mathcal{P} = \{V_1, \ldots, V_l\} \) of \( V(H) \) is called a connected partition if every \( H[V_i] \) \( (1 \leq i \leq l) \) is connected in some color. Similarly, changing partition to cover, we can define connected cover for every edge-colored hypergraph. (Note that, the subsets of the monochromatic connected components of a hypergraph not necessary can be used as parts of a connected partition.) Since partition into vertices is always a connected partition, we can define \( cp(H), cc(H) \) for any edge-colored hypergraph \( H \) as the minimum number of classes in a connected partition or connected cover, respectively. Observe that for trivial hypergraphs \( cc(H) = cp(H) = \alpha(H) = \lvert V(H) \rvert \).

First we will prove the following statement on coverings.

**Theorem 2.** For any 2-edge-colored hypergraph \( H \), we have \( cc(H) \leq \alpha_1(H) \).

In fact, the benefit of introducing the concept of \( \alpha_1(H) \) is to provide an upper bound on \( cc(H) \) in terms of \( \alpha(H) \). From Proposition 1 one also gets the following important corollary:

**Corollary 3.** For any 2-edge-colored non-trivial \( r \)-uniform hypergraph \( H \), we have \( cc(H) \leq \alpha(H) - r + 2 \).

One of our main results is the strengthening of Corollary 3 for partitions.

**Theorem 4.** For any 2-edge-colored non-trivial \( r \)-uniform hypergraph \( H \), we have \( cp(H) \leq \alpha(H) - r + 2 \).

The previous results are sharp. To see this, consider the union of one complete \( r \)-uniform hypergraph and several isolated vertices. Observe that, the partition version of Theorem 2 does not hold. For example, for the hypergraph \( H \) having two edges of size \( r \) intersecting in one vertex, one red and one blue, we have \( cc(H) = 2 \) and \( cp(H) = r (= \alpha(H) - r + 2) \).

It is worth noting that for \( r = 2 \) Theorem 4 extends the \( k = 2 \) case of Conjecture 1. Now we have the following general property for 2-edge-colored graphs.

**Theorem 5.** Any 2-edge-colored graph \( G \) can be partitioned into \( \alpha(G) \) monochromatic connected parts.

An edge-coloring of a graph is called a Gallai-coloring if there is no rainbow triangle in it, i.e. every triangle is colored by at most two colors. Gallai-colorings are natural extensions of
2-colorings and have been recently investigated in many papers (for references see [6]). It is known that, any Gallai-colored complete graph has a monochromatic spanning tree (see e.g. [7]). So we have \( cp(G) = cc(G) = 1 \) if \( G \) is a Gallai-colored complete graph. Now we focus on Gallai-colored general graphs. Our result is the following:

**Theorem 6.** Let \( G \) be a Gallai-colored graph with \( \alpha(G) = \alpha \). Then, with a suitable function \( g(\alpha) \), we have \( cp(G) \leq g(\alpha) \).

Theorem 6 extends the result proved by Gyárfás, Simonyi and Tóth [8] that in any Gallai coloring of a graph \( G \), \( cc(G) \) is bounded in terms of \( \alpha(G) \). We shall also improve on a result in [8] about dominating sets of multipartite digraphs.

## 2 Partitions of 2-edge-colored hypergraphs, proof of Theorem 4

We first prove the cover version.

**Proof of Theorem 2.** Let \( H \) be a hypergraph 2-edge-colored with red and blue. For every vertex \( v \in V(H) \) let \( R(v), B(v) \) denote the monochromatic connected components containing \( v \) in the hypergraphs of the red and blue edges, respectively. (One or both can be a single component containing \( v \).)

From \( H \) we construct a bipartite graph \( G \) with bipartition \( V(G) = (R, B) \), where \( R = \{ R(v) | v \in V(H) \} \), \( B = \{ B(v) | v \in V(H) \} \) and with edge set \( E(G) = \{ R(v)B(v) | v \in V(H) \} \). By the construction, note that \( |E(G)| = |V(H)| \) and \( G \) may contain multiple edges. Also we can regard an edge in \( E(G) \) as a vertex in \( H \).

Notice that for any two independent edges \( e = R(v)B(v), e' = R(u)B(u) \in E(G) \), there is no monochromatic connected component containing \( v \) and \( u \), and hence there is no edge in \( H \) containing both \( v \) and \( u \). Therefore the maximum number of independent edges in \( G \), \( \nu(G) \), satisfies \( \nu(G) \leq \alpha_1(H) \).

By König’s theorem, the edges of \( G \) have a transversal of \( \nu(G) \) vertices, i.e., there is a subset \( T \subseteq V(G) \) such that \( |T| = \nu(G) \) and \( T \) intersects all edges of \( G \) in at least one vertex. Then the monochromatic components of \( H \) corresponding to the vertices of \( T \) form a desired covering of \( V(H) \).

**Remark.** Conjecture 1 for \( k = 2 \) (its proof is implicitely in [5, 7]) implies Theorem 2 directly as follows. The cover graph \( G_H \) of \( H \) can be covered by \( \alpha(G_H) = \alpha_1(H) \) monochromatic connected components and so \( cc(H) \leq \alpha_1(H) \) also holds.

Next, we turn to the proof of the partition version.

**Proof of Theorem 4.** Let \( H \) be a non-trivial \( r \)-uniform hypergraph with independence number \( \alpha(H) \). The proof goes by induction on \( \alpha(H) \). In the base case, when \( \alpha(H) = r - 1 \), i.e.
$H$ is a 2-edge-colored complete $r$-uniform hypergraph, it follows from Corollary 3 that one monochromatic component covers the vertices.

Suppose $\alpha(H) > r - 1$. By Corollary 3, $V(H)$ can be covered by the vertices of $p$ red components, $R_1, \ldots, R_p$, and $q$ blue components, $B_1, \ldots, B_q$, so that

$$p + q \leq \alpha(H) - r + 2. \quad (1)$$

We may assume that $p, q$ are both positive, since if one of them is zero, we already have the desired partition in the other color. Set $R = (\bigcup_{1 \leq i \leq p} R_i) \setminus (\bigcup_{1 \leq i \leq q} B_i)$ and $B = (\bigcup_{1 \leq i \leq q} B_i) \setminus (\bigcup_{1 \leq i \leq p} R_i)$. If $R$ or $B$ is empty, we have again the required partition. Thus we may assume that both $R$ and $B$ are non-empty, so $\alpha(H[R]) \geq 1$, and $\alpha(H[B]) \geq 1$. Observe that both $R(B)$ and $B(R)$ are non-empty, so $\alpha(H[R]) = 1$, and $\alpha(H[R]) = 1$. Observe that $\alpha(H[R]) = 1$ and $\alpha(H[R]) = 1$. If $H[R]$ is non-trivial, then $\alpha(H[R]) = \alpha(H[R]) - r + 2$ by the inductive hypothesis, but if $H[R]$ is trivial then $\alpha(H[R]) = \alpha(H[R])$. Similarly, if $H[B]$ is non-trivial, then $\alpha(H[B]) = \alpha(H[B]) - r + 2$, if $H[B]$ is trivial then $\alpha(H[B]) = \alpha(H[B])$.

**Case 1.** $H[R]$ is non-trivial (and $H[B]$ is either non-trivial or trivial).

Thus $R$ (the vertex set of $H[R]$) has a connected partition $\mathcal{P}_R$ into at most $\alpha(H[R]) - r + 2$ parts. The set $B$ (the vertex set of $H[B]$) has a connected partition $\mathcal{P}_B$ into at most $\alpha(H[B])$ parts. Hence $\mathcal{P}_R \cup \{B_1, \ldots, B_q\}$ and $\mathcal{P}_B \cup \{R_1, \ldots, R_p\}$ are two connected partitions on $V(H)$. Using (1),(2) we have

$$(|\mathcal{P}_R| + q) + (|\mathcal{P}_B| + p) \leq (\alpha(H[R]) - r + 2) + \alpha(H[B]) + p + q \leq 2(\alpha(H) - r + 2),$$

therefore one of the previous connected partitions has at most $\alpha(H) - r + 2$ parts, as desired.

The case when $H[B]$ is non-trivial goes similarly.

**Case 2.** $H[R]$ and $H[B]$ are both trivial.

Assume $p \geq q$, and select a vertex $v$ from $R$, without loss of generality $v \in R_p$. Observe that no blue edge contains $v$, because $H[R]$ is trivial. Hence every edge containing $v$ is in $R_p$, implying that $\alpha(H \setminus R_p) \leq \alpha(H) - 1$. If $p > 1$ then $H \setminus R_p$ is non-trivial, thus by induction $H \setminus R_p$ has a connected partition with at most $(\alpha(H) - 1) - r + 2$ parts, adding $R_p$ we obtain the required partition for $H$. We conclude $p = q = 1$.

Let $S$ be a maximal (non-extendable) independent set of $H$ in the form $R \cup B \cup M$. By definition of $S$ (and as $H$ is non-trivial) there exist a hyperedge intersecting $M \cup R$ or $M \cup B$ in exactly $r - 1$ vertices (since no edge can intersect both $R$ and $B$), assume the former. Therefore $r \leq |M| + |R| + 1$, this yields

$$\alpha(H) - r + 2 \geq |S| - r + 2 = |R| + |B| + |M| - r + 2 \geq |R| + |B| + |M| - (|M| + |R| + 1) + 2 = |B| + 1,$$

thus the red component, $R_1$ and vertices of $B$ gives a partition of $H$ into at most $\alpha(H) - r + 2$ connected parts. \qed
3 Partitions of Gallai-colored graphs, proof of Theorem 6

We need some notions introduced in [8]. If D is a digraph and \( U \subseteq V(D) \) is a subset of its vertex set then \( N_+(U) = \{ v \in V(D) | \exists u \in U \ (u, v) \in E(D) \} \) is the outneighborhood of \( U \). The closed outneighborhood \( N_+(U) \) of \( U \) is meant to be the set \( U \cup N_+(U) \). A multipartite digraph is a digraph \( D \) whose vertices are partitioned into classes \( A_1, \ldots, A_t \) of independent vertices. Let \( S \subseteq [t] \). A set \( U = \bigcup_{i \in S} A_i \) is called a dominating set of size \( |S| \) if for any vertex \( v \in \bigcup_{i \in S} A_i \) there is a \( w \in U \) such that \( (w, v) \in E(D) \). The smallest \( |S| \) for which a multipartite digraph \( D \) has a dominating set \( U = \bigcup_{i \in S} A_i \) is denoted by \( k(D) \). Let \( \beta(D) \) be the cardinality of the largest independent set of \( D \) whose vertices are from different partite classes of \( D \). (We sometimes refer to them as transversal independent sets.) An important special case is when \( |A_i| = 1 \) for each \( i \in [t] \). Then it follows that \( \beta(D) = \alpha(D) \) and \( k(D) = \gamma(D) \), the usual domination number of \( D \), the smallest number of vertices in \( D \) whose vertices are partitioned into classes \( A_1, \ldots, A_t \) of independent vertices.

Theorem 7 ([8]). Suppose that \( D \) is a multipartite digraph such that \( D \) has no cyclic triangle. If \( \beta(D) = 1 \) then \( k(D) = 1 \) and if \( \beta(D) = 2 \) then \( k(D) \leq 4 \).

Theorem 8 ([8]). For every integer \( \beta \) there exists an integer \( h = h(\beta) \) such that the following holds. If \( D \) is a multipartite digraph without cyclic triangles and \( \beta(D) = \beta \), then \( k(D) \leq h \).

To keep the paper self-contained we give a proof for this statement with a slightly better bound than the one presented in [8].

Proof of Theorem 8. Set \( h(1) = 1, h(2) = 4 \) and \( h(\beta) = \beta + (\beta + 1)h(\beta - 1) \) for \( \beta \geq 3 \). The proof goes by induction on \( \beta \). By Theorem 7, we may assume that \( \beta \geq 3 \) and the theorem is proved for \( \beta - 1 \). Let \( D \) be a multipartite digraph with no cyclic triangle and \( \beta(D) = \beta \). For each \( x \in V(D) \), let \( Z^{(x)} \) be the partite class containing \( x \). Let \( k_1, \ldots, k_\beta \) be \( \beta \) vertices of \( D \), each from a different partite class, such that \( |N_+(\{k_1, \ldots, k_\beta\}) \cup (\bigcup_{1 \leq i \leq \beta} Z^{(k_i)})| \) is maximal.

Let \( K_1 = \{Z^{(k_i)} | 1 \leq i \leq \beta \} \). For each partite class \( Z \not\in K_1 \), let \( Z_0 = Z \cap N_+(\bigcup_{1 \leq i \leq \beta} Z^{(k_i)}) \). For every \( i \) with \( 1 \leq i \leq \beta \), let \( Z_i \) be the set of vertices in \( Z - Z_0 \) that are not sending an edge to \( k_i \), but sending an edge to \( k_j \) for all \( j < i \). Finally, let \( Z_{\beta+1} \) denote the remaining part of \( Z \), the set of those vertices of \( Z \) that does not belong to \( N_+(\bigcup_{1 \leq i \leq \beta} Z^{(k_i)}) \) and send an edge to all vertices \( k_1, \ldots, k_\beta \). (We will refer to the set \( Z_i \) as the \( i \)-th part of \( Z \).) The subgraph \( D_i \) of \( D \) induced by the \( i \)-th parts of the partite classes of \( D - (\bigcup_{1 \leq i \leq \beta} Z^{(k_i)}) \) is also a multipartite digraph with no cyclic triangle. For every \( i \) with \( 1 \leq i \leq \beta \), since adding \( k_i \) to any transversal independent set of \( D_i \) we get a larger transversal independent set, it satisfies \( \beta(D_i) \leq \beta - 1 \).

Suppose that \( \beta(D_{\beta+1}) \geq \beta \). Let \( \{l_1, \ldots, l_\beta\} \) be a transversal independent set of \( D_{\beta+1} \).

Claim. For every \( x \in (N_+(\{k_1, \ldots, k_\beta\}) \cup (\bigcup_{1 \leq i \leq \beta} Z^{(k_i)})) - (\bigcup_{1 \leq i \leq \beta} Z^{(k_i)}) \), we have \( x \in N_+(\{l_1, \ldots, l_\beta\}) \).
Proof. Suppose that \( x \in N_+(\{k_1, \ldots, k_\beta\}) - \bigcup_{1 \leq i \leq \beta} Z^{(i)} \). Then there exists an integer \( 1 \leq i_0 \leq \beta \) such that \((k_{i_0}, x) \in E(D)\). Recall that \((l_i, k_{i_0}) \in E(D)\) for every \( 1 \leq i \leq \beta \). Since \( \{x, l_1, \ldots, l_\beta\} \) is not independent and \( D \) has no cyclic triangle, \( x \in N_+(\{l_1, \ldots, l_\beta\}) \), as desired. Thus we may assume that \( x \in \bigcup_{1 \leq i \leq \beta} Z^{(k_i)} \). Recall that \((x, l_i) \notin E(D)\) for every \( 1 \leq i \leq \beta \). Since \( \{x, l_1, \ldots, l_\beta\} \) is not independent, \( x \in N_+(\{l_1, \ldots, l_\beta\}) \).

Thus we have \( N_+(\{k_1, \ldots, k_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(k_i)} \subseteq N_+(\{l_1, \ldots, l_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(l_i)} \). Since \( l_1 \in (N_+(\{l_1, \ldots, l_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(l_i)}) - (N_+(\{k_1, \ldots, k_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(k_i)}) \), it follows

\[
\left| N_+(\{k_1, \ldots, k_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(k_i)} \right| \leq \left| N_+(\{l_1, \ldots, l_\beta\}) \cup \bigcup_{1 \leq i \leq \beta} Z^{(l_i)} \right|,
\]

which contradicts the choice of \( k_1, \ldots, k_\beta \). Thus \( \beta(D_{i+1}) \leq \beta - 1 \).

By induction on \( \beta \), \( D_i \ (1 \leq i \leq \beta + 1) \) can be dominated by at most \( h(\beta - 1) \) partite classes. Let \( K_2 \) be the appropriate \((\beta + 1)h(\beta - 1)\) partite classes such that \( \bigcup_{Z \in K_2} Z \) dominates \( \bigcup_{1 \leq i \leq \beta + 1} V(D_i) \). Hence we constructed a dominating set \( \bigcup_{Z \in K_1 \cup K_2} Z \) of \( D \) containing at least \( \beta + (\beta + 1)h(\beta - 1) \) partite classes.

This completes the proof of Theorem 8.

To prepare the proof of Theorem 6 we need the following lemma about trees.

**Lemma 9.** Let \( t \geq 1 \) be an integer. Let \( T \) be a tree of order at least \( t \). Then there exist two set \( R \subseteq C \subseteq V(T) \) such that \(|R| = t\), \(|C| \leq 2t\), \( T[C] \) is connected, and either \( T - R \) is connected or \( V(T) = R \).

**Proof.** If \(|V(T)| = t\), then the lemma holds by choosing \( R = C = V(T) \). Thus we may assume that \(|V(T)| \geq t+1\). For each edge \( xy \in E(T) \), let \( T^x_y \) denote the component of \( T \) containing \( x \). Note that \(|\{x\} \cup (\bigcup_{y \in N(x)} V(T^y_{xy}))| = |V(T)| \geq t+1\) for every \( x \in V(T) \). We choose a vertex \( x_0 \in V(T) \) and a subset \( A_0 \subseteq N(x_0) \) such that

(i) \(|\{x_0\} \cup (\bigcup_{y \in A_0} V(T^y_{x_0}))| \geq t+1\), and

(ii) subject to (i), \(|\{x_0\} \cup (\bigcup_{y \in A_0} V(T^y_{x_0}))|\) is minimized.

By the definition of \( x_0 \) and \( A_0 \), we have \( A_0 \neq \emptyset \). Set \( a = |\{x_0\} \cup (\bigcup_{y \in A_0} V(T^y_{x_0}))| \).

**Claim.** \( a \leq 2t \).

**Proof.** Suppose that \( a \geq 2t+1 \). If \(|A_0| = 1\), say \( A_0 = \{y_0\} \), then \(|\{y_0\} \cup (\bigcup_{y \in N(y_0) \setminus \{x_0\}} V(T^y_{y_0x_0}))| = a - 1 \geq t+1\), which contradicts the definition of \( x_0 \) and \( A_0 \). Thus \(|A_0| \geq 2\). Then there exists a vertex \( y_1 \in A_0 \) such that \(|V(T^y_{x_0y_1})| \leq (a-1)/2\). Hence

\[
|\{x_0\} \cup \left( \bigcup_{y \in A_0 \setminus \{y_1\}} V(T^y_{x_0y_1}) \right)| = a - |V(T^y_{x_0y_1})| \geq a - \frac{a-1}{2} = \frac{a+1}{2} \geq \frac{2t+2}{2} = t+1,
\]
which contradicts the definition of $A_0$. \hfill \Box

Write $\bigcup_{y \in A_0} V(T_{xy}) = \{x_1, \ldots, x_{a-1}\}$, we may assume that the elements of this set are ordered in a non-increasing order by the distance from $x_0$. Let $C = \{x_0\} \cup (\bigcup_{y \in A_0} V(T_{xy}))$ and $R = \{x_i \mid 1 \leq i \leq t\}$. Then $|R| = t$, $|C| \leq 2t$ and both $T|C|$ and $T - R$ are connected. \hfill \Box

Now we are ready to prove Theorem 6. Let $g(1) = 1$ and $g(\alpha) = \max\{h(\alpha)(\alpha^2 + \alpha - 1), 2h(\alpha)g(\alpha - 1) + h(\alpha) + 1\}$ for $\alpha \geq 2$.

**Proof of Theorem 6.** We show that $\text{cp}(G) \leq g(\alpha(G))$ with the function $g$ defined above. We may assume that $|V(G)| \geq g(\alpha)$. We proceed by induction on $\alpha$. If $\alpha = 1$, then $G$ is complete, and hence there is a connected monochromatic spanning subgraph of $G$, as desired. Thus we may assume that $\alpha \geq 2$. Let $T_0$ be a maximum connected spanning monochromatic subtree of $G$ in the coloring $c$. We may assume that every edge of $T_0$ has color 1. It was proved in [7] that the largest monochromatic subtree in every Gallai-coloring of a graph $G$ has at least $|V(G)|(\alpha^2 + \alpha - 1)^{-1}$ vertices. Using this, since $|V(G)| \geq g(\alpha) \geq h(\alpha)(\alpha^2 + \alpha - 1)$, $|V(T_0)| \geq h(\alpha)$ follows. By Lemma 9, there exist two sets $R$ and $C$ with $R \subseteq C \subseteq V(T_0)$ such that $|R| = h(\alpha)$, $|C| \leq 2h(\alpha)$, $T_0[C]$ is connected, and either $T_0 - R$ is connected or $V(T_0) = R$. Write $C = \{u_1, \ldots, u_m\}$. Note that $h(\alpha) \leq m \leq 2h(\alpha)$. We may assume that $R = \{u_1, \ldots, u_{h(\alpha)}\}$. For every $i$ with $1 \leq i \leq m$, let $U_i$ be the set of vertices in $V(G) - V(T_0)$ that are not adjacent to $u_i$, but adjacent to $u_j$ for all $j < i$. For every $i$ with $1 \leq i \leq m$, we have $\alpha(G[U_i]) \leq \alpha - 1$ because adding $u_i$ to any independent set of $G[U_i]$ we get a larger independent set. By the inductive assumption, for every $i$ with $1 \leq i \leq m$, there exists a partition $\mathcal{P}_i$ of $U_i$ such that $|\mathcal{P}_i| \leq g(\alpha - 1)$ and, for every $U \in \mathcal{P}_i$, $G[U]$ has a connected spanning monochromatic subgraph concerning $c$.

Let $U_0 = V(G) - \left(V(T_0) \cup \left(\bigcup_{1 \leq i \leq m} U_i\right)\right)$. Recall that $T_0[C]$ is a connected monochromatic tree and $c$ is a Gallai-coloring of $G$. For every $v \in U_0$, since $v$ is adjacent to every vertex of $C$, all of $E(v, C)$ are colored with the same color, say $c_v$. Note that $c_v \neq 1$ for every $v \in U_0$ by the definition of $T_0$. Let $l$ be the number of colors used on edges of $E(U_0, C)$. We may assume that $2, \ldots, l + 1$ are the colors used on these edges. For each $i$ with $2 \leq i \leq l + 1$, $A_i = \{v \in U_0 \mid c_v = i\}$. Note that $\{A_2, \ldots, A_{l+1}\}$ is a partition of $U_0$. Since $c$ is a Gallai coloring of $G$, each edge between $A_i$ and $A_j$ is colored with either color $i$ or $j$ for $i,j$ with $2 \leq i,j \leq l + 1$ and $i \neq j$.

We construct the multipartite digraph $D$ on $U_0$ as follows:

(i) $A_2, \ldots, A_{l+1}$ are the partition classes of $D$.

(ii) For $i,j$ with $2 \leq i,j \leq l + 1$ and $i \neq j$, $v \in A_i$ and $v' \in A_j$, let $(v, v') \in E(D)$ if and only if $vv' \in E(G)$ and $c(vv') = i$. 

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Note that \( \beta(D) \leq \alpha \) and \( D \) has no cyclic triangle. By Theorem 8, there exist at most \( h(\alpha) \) partite classes dominating \( V(D) \), say \( B_1, \ldots, B_p \). Let \( B_{p+1} = \cdots = B_{h(\alpha)} = \emptyset \). For every \( i \) with \( 1 \leq i \leq h(\alpha) \), let \( B'_i \) be the set of vertices in \( U_0 - \bigcup_{1 \leq i \leq h(\alpha)} B_i \) that are dominated by \( B_i \), but not dominated by \( B_j \) for all \( j < i \), and let \( B''_i = \{ u_i \} \cup B_i \cup B'_i \). For each \( i \) with \( 1 \leq i \leq h(\alpha) \), note that \( G[B''_i] \) has a connected monochromatic spanning subgraph. Therefore \( \mathcal{P} = \{ V(T_0) - R, B''_1, \ldots, B''_{h(\alpha)} \} \cup \left( \bigcup_{1 \leq i \leq m} P_i \right) \) is a partition of \( V(G) \) satisfying that \( G[U] \) has a connected spanning monochromatic subgraph concerning \( c \) for every \( U \in \mathcal{P} \). Furthermore,

\[
|\mathcal{P}| \leq (h(\alpha) + 1) + \sum_{1 \leq i \leq m} |P_i| \leq (h(\alpha) + 1) + \sum_{1 \leq i \leq m} g(\alpha - 1) = (h(\alpha) + 1) + mg(\alpha - 1) \leq (h(\alpha) + 1) + 2h(\alpha)g(\alpha - 1).
\]

This completes the proof of Theorem 6. \( \square \)

4 Conclusion, open problems

The quantities \( cc(G), cp(G) \) can be far apart, even for 2-edge-colored graphs. For example, let \( G \) be a star with \( 2t \) edges and color \( t \) edges in both colors. Then \( cc(G) = 2, cp(G) = t + 1 \). Nevertheless, the extension of Conjecture 1 to partitions of complete graphs have been formulated in [3]. Probably this remains true for Ryser’s conjecture in general.

**Conjecture 2.** If the edges of \( G \) are colored with \( k \) colors then \( cp(G) \leq \alpha(G)(k - 1) \).

As mentioned before, Conjecture 2 is proved for \( \alpha(G) = 1, k = 3 \) in [3]. Note that \( cc(G) \leq \alpha(G)k \) is obvious for any \( k \)-edge-colored graph \( G \). For \( k \)-edge-colored complete graphs \( K \), Haxell and Kohayakawa [10] proved \( cp(K) \leq k \), this is just one off from Conjecture 2. It would be interesting to attack the case \( k = 3 \) in Conjecture 2 since its cover version, Conjecture 1 is available ([1]).

As mentioned in the introduction, Király [12] solved completely the cover problem for complete \( r \)-uniform complete hypergraphs (\( r \geq 3 \)). (The number of colors \( k \) can be arbitrary.) It seems that the analogue for partition is not easy. A first test case might be the following.

**Problem 3.** Suppose that a complete 3-uniform hypergraph \( H \) is 6-edge-colored. Is it true that \( cp(H) \leq 2 \) ? (\( cc(H) \leq 2 \).)

In general, the cover problem of hypergraphs for general \( \alpha \) or \( \alpha_1 \) seems difficult, even to find the right conjecture is a challenge. We shall address this question in [4].
References


