New Bounds on the Grundy Number of Products of Graphs

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Abstract: The Grundy number of a graph \( G \) is the largest \( k \) such that \( G \) has a greedy \( k \)-coloring, that is, a coloring with \( k \) colors obtained by applying the greedy algorithm according to some ordering of the vertices of \( G \). In this article, we give new bounds on the Grundy number of the product of two graphs.

Keywords: coloring; product; Grundy

1. INTRODUCTION

Graphs considered in this article are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise). The definitions and notation used in this article are standard and may be found in any textbook on graph theory; see [4] for example.

Given two graphs \( G \) and \( H \), the direct product \( G \times H \), the lexicographic product \( G[H] \), the Cartesian product \( G \Box H \), and the strong product \( G \star H \) are the graphs with vertex set \( V(G) \times V(H) \) and the following edge sets:

\[
E(G \times H) = \{(a,x)(b,y) | ab \in E(G) \text{ and } xy \in E(H)\};
\]

\[
E(G[H]) = \{(a,x)(b,y) | \text{ either } ab \in E(G) \text{ or } a = b \text{ and } xy \in E(H)\};
\]

\[
E(G \Box H) = \{(a,x)(b,y) | \text{ either } a = b \text{ and } xy \in E(H) \text{ or } ab \in E(G) \text{ and } x = y\};
\]

\[
E(G \star H) = E(G \times H) \cup E(G \Box H).
\]

A \( k \)-coloring of a graph \( G \) is a surjective mapping \( \psi : V(G) \to \{1, \ldots, k\} \). It is proper if for every edge \( uv \in E(G) \), \( \psi(u) \neq \psi(v) \). A proper \( k \)-coloring may also be seen as a partition of the vertex set of \( G \) into \( k \) disjoint non-empty stable sets (i.e. sets of pairwise non-adjacent vertices) \( C_i = \{v | \psi(v) = i\} \) for \( 1 \leq i \leq k \). For convenience (and with a slight abuse of terminology), by proper \( k \)-coloring we mean either the mapping \( \psi \) or the partition \( \{C_1, \ldots, C_k\} \). The elements of \( \{1, \ldots, k\} \) are called colors. A graph is \( k \)-colorable if it admits a \( k \)-coloring. The chromatic number \( \chi(G) \) is the least \( k \) such that \( G \) is \( k \)-colorable.

Many upper bounds on the chromatic number arise from algorithms that produce colorings. The most basic one is the greedy algorithm. A greedy coloring relative to a vertex ordering \( v_1 < v_2 < \cdots < v_n \) of \( V(G) \) is obtained by coloring the vertices in the order \( v_1, \ldots, v_n \), assigning to \( v_i \) the smallest positive integer not already used on its lower indexed neighbors. Trivially, a greedy coloring is proper. Denoting by \( C_i \) the stable set of vertices colored \( i \), a greedy coloring has the following property:

For every \( i < j \), every vertex in \( C_j \) has a neighbor in \( C_i \) \hspace{1cm} (*)

for otherwise the vertex in \( C_j \) would have been colored \( i \) or less. Conversely, a coloring satisfying Property (\( * \)) is a greedy coloring relative to any vertex ordering in which the vertices of \( C_i \) precede those of \( C_j \) whenever \( i < j \). The Grundy number \( \Gamma(G) \) is the largest \( k \) such that \( G \) has a greedy \( k \)-coloring.
Let $\Delta(G)$ denote the maximum degree in a graph $G$. Let $K_n$ denote the complete graph on $n$ vertices and $K_{p,q}$ denote the complete bipartite graph with parts of size $p$ and $q$. Let $S_n$ denote the edgeless graph on $n$ vertices.

In [1], Asté et al. investigated the Grundy number of several types of graph products. They showed that the Grundy number of the lexicographic product of two graphs is bounded in terms of the Grundy numbers of these graphs.

**Theorem 1** (Asté et al. [1]). For any two graphs $G$ and $H$, $\Gamma(G[H]) \leq 2^{\Gamma(G)} - 1 + \Gamma(G)$.

Moreover, when the graph $G$ is a tree, they obtained an exact value.

**Theorem 2** (Asté et al. [1]). Let $T$ be a tree and $H$ be any graph. Then $\Gamma(T[H]) = \Gamma(T)\Gamma(H)$.

They also showed that, in contrast with the lexicographic product, there is no upper bound of $\Gamma(G\boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$; for example, $\Gamma(K_{p,p}) = 2$ and $\Gamma(K_{p,p} \boxtimes K_{p,p}) \geq p + 1$. Nevertheless, they showed that $\Gamma(G\boxtimes H)$ is bounded by a function of $\Delta(G)$ and $\Gamma(H)$.

**Theorem 3** (Asté et al. [1]). For any two graphs $G$ and $H$, $\Gamma(G\boxtimes H) \leq \Delta(G) \cdot 2^{\Gamma(H) - 1} + \Gamma(H)$.

However, they conjectured that this upper bound is far from being tight.

**Conjecture 4** (Asté et al. [1]). For any two graphs $G$ and $H$, $\Gamma(G\boxtimes H) \leq (\Delta(G) + 1)\Gamma(H)$.

This conjecture generalizes the following conjecture of Balogh et al. [3].

**Conjecture 5** (Balogh et al. [3]). For any graph $H$, $\Gamma(K_2 \boxtimes H) \leq 2^\Gamma(H)$.

Here is another conjecture that would imply the preceding one.

**Conjecture 6** (Havet and Zhu). If $G$ is any graph and $M$ is a matching in $G$, then $\Gamma(G) \leq 2\Gamma(G \setminus M)$.

In [7], Havet et al. proved Conjecture 4 in the case when one of $G, H$ is a tree.

**Theorem 7** (Havet et al. [7]). For any graph $G$ and tree $T$, $\Gamma(G \boxtimes T) \leq (\Delta(G) + 1)\Gamma(T)$.

Here, we investigate further the relation between the Grundy number of the direct product, lexicographic product or Cartesian product of two graphs and the invariants $\Gamma$ and $\Delta$ of the two graphs. We first show that $\Gamma(G\boxtimes H) \leq \Gamma(H|_{K_{\Delta(G) + 1}})$. Together, with Theorems 1 and 2, this implies Theorems 3 and 7, respectively. In particular, we obtain a shorter proof of Theorem 7.

We then show that $\Gamma(G[K_2]) = \Gamma(G[S_2] \boxtimes K_2)$. As a corollary, we give an example of a graph that disproves Conjectures 4–6: there is a graph $H$ such that $\Gamma(H) = 3$ and $\Gamma(K_2 \boxtimes H) = 7$. Together with Theorem 3 this yields $\max \{\Gamma(K_2 \boxtimes H) | \Gamma(H) = 3\} = 7$.

Regarding the direct and strong product, we answer a question raised as the last sentence in [1]. There cannot be any bound on $\Gamma(G \times H)$ and $\Gamma(G \boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$ if $\Gamma(G)$ and $\Gamma(H)$ are both $\geq 3$ (Theorem 15). It is also impossible to
bound $\Gamma(G \times H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when $G$ is any graph with at least one edge and $\Gamma(H) \geq 5$ (Theorems 17). Similarly, it is impossible to bound $\Gamma(G \overline{\square} H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when $\Gamma(H) \geq 5$ unless $G$ is the disjoint union of complete graphs (Theorem 18 and Proposition 19).

## 2. THE CARTESIAN AND LEXICOGRAPHIC PRODUCTS

### A. Common Proof of Theorems 3 and 7

**Theorem 8.** For any two graphs $G$ and $H$, $\Gamma(G \square H) \leq \Gamma(G[K_{\Delta(H)+1}])$.

**Proof.** We shall prove that if $G \square H$ has a greedy $q$-coloring for some integer $q$, then so does $G[K_{\Delta(H)+1}]$. Hence, consider a greedy $q$-coloring $\phi$ of $G \square H$.

Let $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of $G$ such that $\phi(x_1, y) \leq \phi(x_2, y) \leq \cdots \leq \phi(x_n, y)$. Let $z_0, \ldots, z_{\Delta(H)}$ be the vertices of $K_{\Delta(H)+1}$. So every vertex of $G[K_{\Delta(H)+1}]$ is a pair $(x_i, z_j)$ for some $i \in \{1, \ldots, n\}$ and $j \in \{0, 1, \ldots, \Delta(H)\}$. Let $(x, y)$ be a vertex of $G \square H$ with color $q$.

For every $i$ in $\{1, \ldots, n\}$, we assign color $\phi(x_i, y)$ to vertex $(x_i, z_0)$ of $G[K_{\Delta(H)+1}]$. Then for $i = 1$ to $n$, we do the following. Let $L_i$ be the set of all colors $\ell$, with $\ell < \phi(x_i, y)$, that have not been assigned to any neighbor of $(x_i, z_0)$ in $G[K_{\Delta(H)+1}]$. Since $\phi$ is a greedy coloring and color $\phi(x_j, y)$ is assigned to $(x_j, z_0)$ for each $j$, $L_i$ is a subset of $\{\phi(x_i, u)|u \in N(y)\}$. Therefore, $|L_i| \leq \Delta(H)$. Hence, we can assign all the colors of $L_i$ to distinct vertices in $\{(x_i, z)|1 \leq j \leq \Delta(H)\}$.

Let us show that the obtained partial $q$-coloring of $G[K_{\Delta(H)+1}]$ is a greedy coloring. It is proper since colors already assigned to neighbors of $(x_i, z_0)$ are not in $L_i$. In $L_i$ we add every color $\ell < \phi(x_i, z_0)$ such that $(x_i, z_0)$ had no neighbor colored $\ell$ before Step $i$. Hence, after Step $i$, vertex $(x_i, z_0)$ has a neighbor of each color less than $\phi(x_i, y)$. Now every colored vertex $(x_i, z)$ has a color $\ell$ less than $\phi(x_i, y)$. But, by the definition of the lexicographic product, all neighbors of $(x_i, z_0)$, except $(x_i, z)$ itself, are neighbors of $(x_i, z)$. Hence, $(x_i, z)$ has a neighbor of each color less than $\ell$. So the coloring is greedy.

### B. Disproof of Conjecture 4

Asté et al. [1] proved the following:

**Lemma 9** (Asté et al. [1]). For any graph $G$ and any integer $n$, $\Gamma(G[S_n]) = \Gamma(G)$.

Now we prove:

**Theorem 10.** Let $G$ be a graph. Then $\Gamma(G[K_2]) = \Gamma(G[S_2] \square K_2)$.

**Proof.** Let us show that the left-hand side is at most the right-hand side. Consider a greedy coloring $\phi$ of $G[K_2]$. Every vertex $v$ of $G$ corresponds to two adjacent vertices of $G[K_2]$. Let us denote by $\phi_1(v)$ and $\phi_2(v)$ the two distinct colors assigned by $\phi$ to these vertices. In the graph $G[S_2] \square K_2$, every vertex $v$ corresponds to four vertices $a_v$, $b_v$, $a'_v$, and $b'_v$ inducing two edges $a_vb_v$ and $a'_vb'_v$, and so that if $uv$ is any edge of $G$, then $G[S_2] \square K_2$ has all edges between $\{a_u, a'_u\}$ and $\{a_v, a'_v\}$ and all edges between $\{b_u, b'_u\}$ and $\{b_v, b'_v\}$.}


and \( \{b_i, b'_i\} \). Assign color \( \varphi_1(v) \) to \( a_v \) and \( b'_i \) and color \( \varphi_2(v) \) to \( b_v \) and \( a'_i \). Doing this for every vertex, it is easy to check that we obtain a greedy coloring of \( G[S_2] \square K_2 \). Hence, \( \Gamma(G(K_2)) \leq \Gamma(G(S_2) \square K_2) \).

Let us now show that the right-hand side is at most the left-hand side. By Theorem 8, we have \( \Gamma(G(S_2) \square K_2) \leq \Gamma(G(S_2)[K_2]) \). We claim that \( \Gamma(G(S_2)[K_2]) \leq \Gamma(G(K_2)) \). To see this, consider any greedy coloring \( \varphi \) of \( G(S_2)[K_2] \) with \( q \) colors. In \( G(S_2)[K_2] \), every vertex \( v \) of \( G \) corresponds to four vertices \( a_v, b_i, c_v, d_v \) with two edges \( a_i b_v, c_v d_v \), and for every edge \( uv \) of \( G \), there are all edges between \( \{a_u, b_u, c_u, d_u\} \) and \( \{a_v, b_v, c_v, d_v\} \). Suppose that \( \varphi \) assigns at least three different colors in \( \{a_v, b_i, c_v, d_v\} \) for some \( v \), say \( \varphi(a_v) = i, \varphi(b_v) = j, \varphi(c_v) = k \), where, up to symmetry, \( i < j \) and \( k \neq \{i, j\} \). Note that \( b_v \) has no neighbor of color \( k \), because its neighbors are either \( a_v \) or adjacent to \( c_v \). So \( j < k \). At least one color \( h \in \{i, j\} \) is not the color of \( d_v \), so \( c_v \) has no neighbor of color \( h \), a contradiction. So \( \varphi \) uses exactly two colors in \( \{a_v, b_i, c_v, d_v\} \) for every vertex \( v \) of \( G \). It follows that the restriction of \( \varphi \) on the subgraph of \( G[S_2][K_2] \) induced by \( \{a_v, b_i, c_v, d_v\}_{V(G)} \), which is isomorphic to \( G(K_2) \), is a greedy coloring with \( q \) colors. So the claim that \( \Gamma(G(S_2) \square K_2) \leq \Gamma(G(K_2)) \) is established. This completes the proof.

**Remark 11.** Theorem 10 can be generalized in a straightforward manner to the following result: Let \( G \) be any graph and \( p \) be any integer. Then \( \Gamma(G(K_p)) = \Gamma(G(S_p) \square K_p) \).

Theorem 10 implies that Conjectures 4–6 do not hold, as follows.

**Corollary 12.** There is a graph \( H \) such that \( \Gamma(H) = 3 \) and \( \Gamma(K_2 \square H) = 7 \).

**Proof.** Let \( G_3 \) be the graph that consists of a cycle of length 6 plus one vertex \( g \) adjacent to a vertex \( a \) of the cycle and one vertex \( h \) adjacent to another vertex \( b \) of the cycle, where \( a \) and \( b \) are adjacent. Let \( H = G_3[S_2] \). Asté et al. [1] showed that \( \Gamma(G_3) = 3 \) and \( \Gamma(G_3[K_2]) = 7 \). Hence, Lemma 9 yields \( \Gamma(H) = 3 \) and Theorem 10 yields \( \Gamma(K_2 \square H) = 7 \). This proves the corollary.

Alternately, let \( G_3' \) be the graph obtained from \( G_3 \) by identifying the two vertices \( g \) and \( h \) (i.e., replacing them by one vertex adjacent to \( a \) and \( b \)), and let \( H' = G_3'[S_2] \). Then one can also check that \( \Gamma(H') = 3 \) and \( \Gamma(K_2 \square H') = 7 \).

Clearly, the two graphs \( H \) and \( H' \) mentioned in the preceding proof are counterexamples to Conjectures 4 and 5. Note also that if \( v \) is any vertex of \( H \) and \( a_v, b_v \) are the corresponding two vertices in \( K_2 \square H \), then the set \( M = \{a_v, b_v\}_{V(H)} \) is a matching in \( K_2 \square H \), and \( (K_2 \square H) \setminus M \) consists of two disjoint copies of \( H \) with no edge between them; so \( \Gamma((K_2 \square H) \setminus M) = 3 \). This shows that \( K_2 \square H \) is a counterexample to Conjecture 6. The same holds for \( K_2 \square H' \).

Corollary 12 shows that Conjecture 4 does not hold if \( \Gamma(H) = 3 \). On the other hand, we now show that Conjecture 4 holds if \( \Gamma(H) = 2 \).

**Proposition 13.** Let \( G \) and \( H \) be two graphs. If \( \Gamma(H) = 2 \), then \( \Gamma(G \square H) \leq 2(\Delta(G) + 1) \).

**Proof.** If \( H \) is not connected, and has components \( H_1, \ldots, H_p \) \( (p \geq 2) \), then \( G \square H \) is the disjoint union of \( G \square H_1, \ldots, G \square H_p \), and it suffices to prove the proposition for each graph \( G \square H_i \). Therefore, we may assume that \( H \) is connected. If \( \Gamma(H) = 2 \), then \( H \) is a complete bipartite graph [10]. Let \( (A, B) \) be its bipartition. For every vertex \( v \in V(G), \)
define \( A_v = \{(v,a) | a \in A\} \) and \( B_v = \{(v,b) | b \in B\} \), so \( A_v \) and \( B_v \) are the two sides of the copy of \( H \) indexed by \( v \) in \( G \square H \). Let \( \varphi \) be a greedy coloring of \( G \square H \). We claim that:

\[
\text{For any } v \text{ in } V(G), \ |\varphi(A_v)| \leq \Delta(G) + 1 \text{ and } |\varphi(B_v)| \leq \Delta(G) + 1.
\]

Assume for a contradiction, and up to symmetry, that \( |\varphi(A_v)| \geq \Delta(G) + 2 \). Let \( x \) be the largest color of \( \varphi(A_v) \) and let \( x = (v, a) \) be a vertex colored \( x \). The neighborhood of \( x \) in \( G \square H \) is \( B_v \cup \{(w, a) | w \in N_G(v)\} \). But the colors of \( \varphi(A_v) \) do not appear on \( B_v \), because it is complete to \( A_v \), and \( \{(w, a) | w \in N_G(v)\} = d_G(v) \leq \Delta(G) \). Hence, at most \( \Delta(G) \) colors of \( \varphi(A_v) \) may appear on the neighborhood of \( x \), and so at least one color of \( \varphi(A_v) \backslash \{x\} \) does not. This contradicts the fact that \( \varphi \) is a greedy coloring and proves the claim.

Let \( y = (v, b) \) be a vertex such that \( \varphi(y) \) is maximum. Without loss of generality, we may assume that \( b \in B \). At most \( 2\Delta(G) + 1 \) colors appear in the neighborhood of \( y \); at most \( \Delta(G) + 1 \) on \( A_v \), according to the claim, and at most one more for each of its neighbors not in \( B_v \), whose number is \( d_G(y) \leq \Delta(G) \). Hence, \( \varphi(y) \leq 2\Delta(G) + 2 \).

**Remark 14.** Proposition 13 can easily be generalized to complete multipartite graphs in a straightforward manner to obtain the following result: if \( H \) is a complete multipartite graph, then \( \Gamma(G \square H) \leq (\Delta(G) + 1) \Gamma(H) \).

### 3. THE DIRECT AND STRONG PRODUCTS

Here we show that \( \Gamma(G \times H) \) and \( \Gamma(G \square H) \) cannot be bounded by a function of \( \Gamma(G) \) and \( \Gamma(H) \) if \( \Gamma(G), \Gamma(H) \geq 3 \) (Theorem 15). It is also a natural question to bound \( \Gamma(G \times H) \) or \( \Gamma(G \square H) \) in terms of \( \Delta(G) \) and \( \Gamma(H) \). For \( \Delta(G) = 1 \), a non-trivial construction of [2] shows that \( 3|\Gamma(H)/2| - 1 \leq \Gamma(K_2 \times H) \). Somewhat surprisingly, we show in Theorem 17 that there is no upper bound on \( \Gamma(K_2 \times H) \) in terms of \( \Gamma(H) \) if \( \Gamma(H) \geq 5 \). Moreover, we show in Theorem 18 that there is no upper bound on \( \Gamma(P_3 \square H) \) in terms of \( \Gamma(H) \) if \( \Gamma(H) \geq 5 \). In fact, Theorem 18 implies that there is no upper bound on \( \Gamma(G \square H) \) as a function \( \Delta(G) \) and \( \Gamma(H) \) for \( \Gamma(H) \geq 5 \) unless \( G \) is the disjoint union of complete graphs. In Proposition 19, we show that there is an upper bound in such a case.

Let us first recall some definitions. The **binomial tree** is the graph \( T_k \) defined recursively as follows. For \( k = 1 \), \( T_1 \) is the one-vertex graph. For \( k \geq 2 \), \( T_k \) is obtained from \( T_{k-1} \) by adding, for each vertex \( v \) of \( T_{k-1} \), one vertex \( v' \) with an edge \( vv' \). It is easy to see that, for \( k \geq 2 \), \( T_k \) has two adjacent vertices \( r, s \) of degree \( k - 1 \) and the other vertices have degree at most \( k - 2 \), and the two components of \( T_k \setminus rs \) are both isomorphic to \( T_{k-1} \). We view \( T_k \) as rooted at vertex \( r \). We have \( \Gamma(T_k) = k \). More precisely, \( T_k \) has a greedy coloring \( \psi \) where each vertex \( v \notin \{r, s\} \) has color equal to its degree, and \( s, r \) have color \( k - 1 \) and \( k \), respectively. Note that for each vertex \( v \) and color \( i \), \( \psi(v) \) has a unique neighbor of color \( i \).

The **radius** of a graph \( G \) is the smallest integer \( t \) for which there exists a vertex \( a \) of \( G \) such that every vertex of \( G \) is at distance at most \( t \) from \( a \). Note that the radius of \( T_k \) is \( k - 1 \). It is easy to see that every tree with radius at most 2 has Grundy number at most 3. This is also a corollary of the following result from [5, 6]: the Grundy number of a tree is equal to the Grundy number of its largest binomial subtree, and of the fact that the radius of a subtree of a tree \( T \) is not larger than the radius of \( T \).
Theorem 15. For every \( k \geq 3 \), there is a graph \( G \) such that \( \Gamma(G) = 3 \) and \( \Gamma(G \times G) \geq k \) and \( \Gamma(G \boxtimes G) \geq k \).

Proof. Let \( G \) be the graph obtained from \( T_k \) by subdividing every edge once. Partition the vertex set of \( G \) into two stable sets \( A \) and \( B \) such that \( A \) contains the original vertices of \( T_k \) and \( B \) contains the subdivision vertices. Consider any greedy coloring of \( G \). Every vertex in \( B \) has degree 2 and consequently receives a color from the set \( \{1,2,3\} \). Moreover, a vertex in \( B \) receives color 3 if and only if its two neighbors have received colors 1 and 2, respectively. It follows that no vertex of \( A \) can receive color 4 or more. This implies that \( \Gamma(G) \leq 3 \). Since \( G \) contains a four-vertex path, \( \Gamma(G) \geq 3 \). Thus \( \Gamma(G) = 3 \).

To complete the proof of the theorem, let us show that \( G \times G \) and \( G \boxtimes G \) have a common induced subgraph \( H_k \) isomorphic to \( T_k \). This implies \( \Gamma(G \times G) \geq k \) and \( \Gamma(G \boxtimes G) \geq k \).

Let the root \( r \) of \( T_k \) become the root of \( G \). Since \( G \) is viewed as a rooted tree, every vertex in \( B \) has one parent and one child. Consider the greedy coloring \( \psi \) of \( T_k \) with \( k \) colors as defined above, such that the root \( r \) has color \( k \) and the second vertex \( s \) of degree \( k-1 \) has color \( k-1 \). For \( i \in \{1,\ldots,k\} \), let \( A_i \) be the set of vertices in \( A \) that receive color \( (k+1)-i \). So \( A_1 = \{ r \} \) and \( A_2 = \{ s \} \). For each \( i \in \{2,\ldots,k\} \), let \( B_i \) be the set of vertices in \( B \) whose child is in \( A_i \). We say that a vertex \( v \) in \( A_i \cup B_i \) has label \( i \), and denote by \( \ell_v \) the label of \( v \). Let \( q \) be the parent of \( s \) (i.e., \( q \) is the common neighbor of \( r \) and \( s \)). Let \( d(x,y) \) denote the distance between any two vertices \( x \) and \( y \) in \( G \). We prove by induction on \( i \in \{2,\ldots,k\} \) that \( G \times G \) and \( G \boxtimes G \) have an induced subgraph \( H_i \) such that:

1. \( H_i \) is isomorphic to \( T_i \) and contains vertex \((r,q)\).
2. Every vertex of \( H_i \) is of the form \((a,b)\) or \((b,a)\), with \( a \in A \) and \( b \in B \); moreover, \( \ell_a < \ell_b \leq i \), vertices \( a \), \( b \) lie in distinct components of \( G \setminus qr \), and \( d(a,r) = d(b,q) \).

For \( i = 2 \), the induced subgraph \( H_2 \) with vertices \((r,q)\) and \((q,r)\) and an edge between them is the desired copy of \( T_2 \). Now let \( i \geq 3 \). By the induction hypothesis, there exists a common induced subgraph \( H_{i-1} \) of \( G \times G \) and \( G \boxtimes G \) that satisfies (1) and (2). Let \( z \) be any vertex of \( H_{i-1} \), and let \( a \in A \) and \( b \in B \) be such that \( z \) is equal to \((a,b)\) or \((b,a)\). Let \( u \) be the unique child of \( b \) in \( G \). By the definition of the labels, we have \( \ell_u = \ell_b \).

By property (2), we have \( \ell_u \leq i - 1 \), so (in \( T_k \), and since \( \psi \) is a greedy coloring) \( a \) has a neighbor of color \((k+1)-i\), and in \( G \) \( a \) has a neighbor \( v \in B \) with label \( i \). Clearly, \( u \) and \( v \) lie in distinct components of \( G \setminus qr \) since \( a \) and \( b \) do. Now, either \((v,u)\) or \((u,v)\) is a neighbor of \((a,b)\) in \( G \times G \) and we call this neighbor the leaf of \( z \), and \( z \) is called the support of its leaf. Note that any leaf-support edge is also an edge in \( G \boxtimes G \) as \( E(G \times G) \subseteq E(G \boxtimes G) \). Since \( v \) has label \( i \), the leaf of \( z \) is not a vertex in \( H_{i-1} \). Since \( \ell_u = \ell_b \leq i - 1 \) and \( \ell_v = i \), we have \( \ell_u < \ell_v \leq i \). Since \( u \) is a child of \( b \) and \( v \) is a child of \( a \), we have \( d(u,r) = d(v,q) \). (More precisely: if \( a \) lies in the component \( G_r \) of \( G \setminus qr \) that contains \( r \) and \( b \) lies in the other component \( G_q \), then \( d(u,r) = d(b,q) + 2 \) and \( d(v,q) = d(a,r) + 2 \); if on the contrary \( a \) lies in \( G_q \) and \( b \) lies in \( G_r \), then \( d(u,r) = d(b,q) \) and \( d(v,q) = d(a,r) \).)

Let \( V_{i-1} \) be the vertex set of \( H_{i-1} \) and let \( W_{i-1} \) be the set of leaves of vertices in \( V_{i-1} \). Let \( H_i \) be the subgraph of \( G \boxtimes G \) induced by the vertices in \( V_{i-1} \cup W_{i-1} \). As observed above, \( H_i \) satisfies property (2). In order to show that \( H_i \) is isomorphic to \( T_i \), we need only to prove that (i) each vertex in \( W_{i-1} \) has a unique neighbor in \( V_{i-1} \) and (ii) \( W_{i-1} \) induces a stable set. Note that this also implies that \( H_i \) is an induced subgraph in \( G \times G \) as \( E(G \times G) \subseteq E(G \boxtimes G) \).

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To show that Claim (i) is true, suppose on the contrary that the leaf \((v,u)\in W_{i-1}\) of some vertex \((a,b)\in V_{i-1}\) is adjacent to a vertex \((x,y)\in V_{i-1}\) different from \((a,b)\). Up to symmetry we may assume that \(a,u\in A\) and \(b,v\in B\) and that \(a\) lies in \(G_r\) and \(b\) in \(G_g\) (the argument in the other cases is similar). We must have \(x=a\), for otherwise \(x\) is either \(v\) or the child of \(v\) and \(\ell_x=i\), which contradicts property (2) in \(H_{i-1}\). Since \(x=a\), then \(y\in B\) by property (2). Now, \(y\neq b\), and \(y\) is a child of \(u\). Now \(d(y,q)=d(b,q)+2\), whereas \(d(x,r)=d(a,r)\), so \(d(x,r)\neq d(y,q)\), a contradiction.

To show that Claim (ii) is true, suppose on the contrary that \((a,b)\) and \((b',a')\) are two adjacent vertices in \(W_{i-1}\). We can consider \(a,a'\in A\) and \(b,b'\in B\) as they could not be adjacent otherwise. Let \((s_a,s_b)\) and \((s'_b,s'_a)\) be the supports of \((a,b)\) and \((b',a')\), respectively. Note that \(s_a,s'_b\in B\) and \(s_b,s'_a\in A\), which implies that \(\ell_{s_b}\ell_{s_a}\ell_{s'_a}\ell_{s'_b}\). By the definition of the labels, we have \(\ell_{s_a}=\ell_{s'_b}\) and \(\ell_{s'_a}=\ell_{s_b}\). Moreover, each of \(b\) and \(b'\) has label \(i\) and consequently has a child of label \(i\) and \(\ell_{s_b}\ell_{s_a}\ell_{s'_a}\ell_{s'_b}\). Thus, for \((a,b)\) to be adjacent to \((b',a')\), \(a\) must be the neighbor of \(b'\) with label smaller than \(i\), which is \(s'_a\).

In particular, \(\ell_{s_a}=\ell_{s'_b}\), and, by a symmetric argument, \(\ell_{s'_a}=\ell_{s_b}\). Putting this all together, we obtain that if \((a,b)\) is adjacent to \((b',a')\), then \(\ell_{s_a}=\ell_{s'_b}<\ell_{s'_a}=\ell_{s_b}\ell_{s_a}\ell_{s'_a}\ell_{s'_b}\), which is a contradiction.

To prove Theorems 17 and 18, we study the graph \(H_k\) defined as follows. We start from the binomial tree \(T_k\) whose vertex set is partitioned into three sets \(X_1,X_2,X_3\). The root of \(T_k\) is in \(X_1\). For every \(v\in X_1\cup X_3\), the children of \(v\) are in \(X_2\). For every \(v\in X_2\), the children of \(v\) are placed according to the position of the parent \(w\) of \(v\): if \(w\in X_1\), then the children of \(v\) are in \(X_3\); if \(w\in X_3\), then the children of \(v\) are in \(X_1\). Now \(H_k\) is obtained by adding to \(T_k\) all edges between \(X_1\) and \(X_3\).

**Theorem 16.** For \(k\geq 1\), \(\Gamma(H_k)\leq 5\). Furthermore, for \(k\geq 9\), \(\Gamma(H_k)=5\).

**Proof.** We first observe that \(\Gamma(H_k)\leq 6\) for every \(k\). Indeed, in \(H_k\) every stable set is contained either in \(A_1=X_1\cup X_2\) or in \(A_2=X_2\cup X_3\). If \(H_k\) admits a greedy coloring with at least seven colors, then at least four color classes are included in one of the two sets \(A_1\) and \(A_2\), say in \(A_j\). This means that the subgraph \(H^*_j\) induced by \(A_j\) in \(H_k\) has Grundy number at least four. However, each component of \(H^*_j\) is a tree of radius at most 2, which implies that \(H^*_j\) has Grundy number at most 3.

In order to complete the first part of the theorem, let us give a more detailed analysis to show that \(\Gamma(H_k)\leq 5\). The following two properties of \(T_k\) are useful:

1. Any vertex \(v\in X_2\) has either exactly one neighbor in \(X_1\) or exactly one neighbor in \(X_3\) (because if the parent of \(v\) is in one of \(X_1\), \(X_3\), then all its children are in the other of these two sets).
2. For \(i=1,3\), no path on five vertices in \(X_i\cup X_2\) has its two endvertices in \(X_i\) (because every component of \(X_i\cup X_2\) consists of either the root of \(T_k\) and its children, or some vertex of \(X_2\), its children and its grandchildren).

Suppose that there exists a greedy 6-coloring \(\varphi\) on \(H_k\).

**Case 1.** \(\varphi(v)\in\{5,6\}\) for \(v\in X_2\). Vertex \(v\) has neighbors of colors 1, 2, 3, 4. By property (1), \(v\) is adjacent to at most one vertex of \(X_1\) or \(X_3\). So there is \(i\in\{1,3\}\) such that \(v\) has neighbors \(w_1,w_2,w_3\in X_i\) with \(\varphi(w_1)\varphi(w_2)\varphi(w_3)\leq 4\). Then \(w_3\) has a neighbor \(w_4\) with \(\varphi(w_4)\varphi(w_2)\varphi(w_3)\leq 4\). Since \(w_2,w_4\) are stable sets, we have \(w_4\in X_2\) and \(w_5\in X_i\). But then the path \(w_1-v-w_3-w_4-w_5\) contradicts property (2).
Case 2. \( \phi(v) = 6 \) for some \( v \in X_1 \cup X_3 \). Let \( i \) be the index in \( \{1, 3\} \) such that \( v \in X_i \). Vertex \( v \) has a neighbor \( w \) with \( \phi(w) = 5 \). Then \( w \in X_{4-i} \), otherwise Case 1 applies. Vertices \( v \) and \( w \) have neighbors \( u_v \) and \( u_w \) of color 4, possibly \( u_v = u_w \), but we cannot have one in \( X_1 \) and the other in \( X_3 \). Hence, one vertex \( u \in \{u_v, u_w\} \) is in \( X_2 \). Let \( t \) be its neighbor in \( \{v, w\} \) and \( j \) the index such that \( t \in X_j \). Vertex \( u \) has three neighbors \( a, b, c \) such that \( \{\phi(a), \phi(b), \phi(c)\} = \{1, 2, 3\} \). By property (1), either two elements of \( \{a, b, c\} \), say \( a, b \), are in \( X_j \), or \( \{a, b, c\} \subseteq X_{4-j} \). If \( a, b \in X_j \), we may assume \( \phi(a) < \phi(b) \), and we pick a neighbor \( d \) of \( t \) with \( \phi(d) = \phi(b) \) and a neighbor \( e \) of \( d \) with \( \phi(e) = \phi(a) \). Since \( \{a, e\} \) and \( \{b, d\} \) are stable sets in \( H_k \), we have \( d \in X_2, e \in X_j \). But then the path \( e \to d \to t \to u \to a \) contradicts property (2). If \( \{a, b, c\} \subseteq X_{4-j} \), we may assume that \( \phi(c) = 1 \), \( \phi(b) = 2 \) and \( \phi(c) = 3 \). There is a neighbor \( d \) of \( c \) with \( \phi(d) = 2 \) and a neighbor \( e \) of \( d \) with \( \phi(e) = 1 \). Since \( \{a, e\} \) and \( \{b, d\} \) are stable sets in \( H_k \), we have \( d \in X_2, e \in X_{4-j} \). But then the path \( e \to d \to c \to u \to a \) contradicts property (2). Thus we have shown that \( \Gamma(H_k) \leq 5 \), which completes the first part of the theorem.

Now, we show that \( \Gamma(H_k) = 5 \) when \( k \geq 9 \). We know that \( \Gamma(T_k) = k \), so \( T_k \) contains a path \( a_1 \to a_2 \to \cdots \to a_9 \) whose vertices are colored \( k, k-1, \ldots, k-8 \), respectively, where \( a_1 \) is the root of \( T_k \), and a path \( a_2 \to b_3 \to b_4 \to b_5 \) whose vertices are colored \( k-1, k-3, k-4, k-5 \), and a path \( a_6 \to b_7 \to b_8 \) whose vertices are colored \( k-5, k-7, k-8 \). Note that vertices \( a_1, a_5, a_9, b_5 \) are in \( X_1 \), vertices \( a_2, a_4, a_6, a_8, b_4, b_8 \) are in \( X_2 \) and vertices \( a_3, a_7, b_3, b_7 \) are in \( X_3 \). Now we can make a greedy coloring of \( H_k \) with five colors, where vertices \( a_2, a_5, a_9, b_5 \) receive color 1, vertices \( a_3, b_3, b_7, a_8 \) receive color 2, vertices \( b_3, a_6 \) receive color 3, and vertices \( a_1 \) and \( a_7 \) receive colors 4 and 5.

Theorem 17. If \( G \) is a graph with at least one edge and \( k \geq 1 \), then \( \Gamma(G \times H_k) \geq k \).

Proof. It is enough to prove the theorem when \( G = K_2 \), \( V(G) = \{v_1, v_2\} \). We claim that \( \Gamma(G \times H_k) \geq k \). To see this, let \( Y_i = \{v_i\} \times X_i \) for \( i = 1, 3 \) and \( Y_2 = \{v_2\} \times X_2 \). Then it is easy to check that \( Y_1 \cup Y_2 \cup Y_3 \) induces a copy of \( T_k \) in \( K_2 \times H_k \), where \( Y_i \) plays the role of \( X_i \) in the partition of \( H_k \).

Theorem 18. If \( G \) is a connected non-complete graph and \( k \geq 1 \), then \( \Gamma(G \boxtimes H_k) \geq k \).

Proof. It is enough to prove the theorem when \( G = P_3 = v_1 \to v_2 \to v_3 \) as \( G \) contains an induced subgraph isomorphic to \( P_3 \). We claim that \( \Gamma(G \boxtimes H_k) \geq k \). To see this, let \( Y_i = \{v_i\} \times X_i \) for \( i = 1, 2, 3 \). It is easy to check that \( Y_1 \cup Y_2 \cup Y_3 \) induces a copy of \( T_k \) in \( P_3 \boxtimes H_k \), where \( Y_i \) plays the role of \( X_i \) in the partition of \( H_k \).

If \( G \) is a the disjoint union of complete graphs, then there is an upper bound on \( \Gamma(G \boxtimes H) \) as a function of \( \Gamma(G) \) and \( \Gamma(H) \). It is enough to consider the case \( G = K_{m+1} \). Observe that \( K_{m+1} \boxtimes H = H[K_{m+1}] \). Hence, by Theorem 1 we get the following.

Proposition 19. If \( \Gamma(H) = k \geq 2 \) and \( m \geq 1 \) then \( \Gamma(K_{m+1} \boxtimes H) \leq m2^{k-1} + k \).

4. COMMENTS AND OPEN QUESTIONS

Section 3 shows that any upper bound on the Grundy number of \( G \times H \) as a function of \( \Delta(G) \), \( \Gamma(H) \) is possible only if \( \Gamma(H) \leq 4 \). Perhaps a good test case is to decide whether
\(\Gamma(K_2 \times H)\) is bounded for \(\Gamma(H) \leq 4\). (On the other hand, if the maximum degree of both graphs may intervene, then we know the easy inequality \(\Gamma(G \times H) \leq \Delta(G \times H) + 1 \leq \Delta(G)\Delta(H) + 1\), but this is probably not a very interesting bound.)

A referee asked whether, in replacement for the now failed Conjecture 4, the following inequality could be conjectured to hold for any two graphs \(G\) and \(H\)

\[
\Gamma(G \Box H) \leq \max((\Delta(G) + 1)\Gamma(H), (\Delta(H) + 1)\Gamma(G))
\]  

(**)

We can prove this inequality, as follows. First, suppose that \(\Gamma(G) = 1\). Then \(G\) has no edge, so the left-hand side of (**) is \(\Gamma(H)\) and the right-hand side is \(\max(\Gamma(H), \Delta(H)+1) = \Delta(H) + 1\), so (**) holds for every \(H\). Now suppose that \(\Gamma(G) \geq 2\) and similarly \(\Gamma(H) \geq 2\). On the right-hand side of (**), we have \(\max((\Delta(G)+1)\Gamma(H), (\Delta(H)+1)\Gamma(G)) \geq \frac{7}{6}(\Delta(G) + 1 + \Delta(H) + 1)\Gamma(G) \geq \Delta(G) + 1 + \Delta(H) + 1\). On the left-hand side, we have \(\Gamma(G \Box H) \leq \Delta(G \Box H) + 1 = \Delta(G) + \Delta(H) + 1\), so it is strictly smaller than the right-hand side. Actually this proof shows that (**) tends to give a weak upper bound on \(\Gamma(G \Box H)\) in general; indeed in all cases it is weaker than \(\Delta(G \Box H) + 1\).

Concerning the lexicographic product, it was proved in [1] that if \(\Gamma(H) = k\), then for any graph \(G\), we have \(\Gamma(G[H]) = \Gamma(G[V[G]])\). Moreover, as mentioned in Remark 11, we have \(\Gamma(G[K_k]) = \Gamma(G[S_k] \Box K_k)\). So \(\Gamma(G[H]) = \Gamma(G[K_k] \Box K_k)\). Thus the Grundy number of the lexicographic product of any two graphs \(G\) and \(H\) can be seen as a particular case of the Grundy number of the Cartesian product of two graphs. Therefore, we feel that the most interesting questions in this domain are about the Cartesian product. In particular, although Conjecture 4 is now known to be false because of Corollary 12, one may still wonder whether there exists a constant \(c\) such that any two graphs \(G\) and \(H\) satisfy \(\Gamma(G \Box H) \leq c(\Delta(G) + 1)\Gamma(H)\). Note that the graph \(H\) given in the proof of Corollary 12 satisfies \(\Gamma(K_2 \Box H) = \frac{7}{6}(\Delta(K_2) + 1)\Gamma(H)\), and so does the second graph \(H'\). We could not find a graph with a ratio larger than \(\frac{7}{6}\). Is it true that \(\Gamma(K_2 \Box H) \leq 2c\Gamma(H)\) for some constant \(c \geq 7/6\)?

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