Around a biclique cover conjecture

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December 31, 2012

Abstract

We address an old (1977) conjecture of a subset of the authors (a variant of Ryser's conjecture): in every r-coloring of the edges of a biclique $[A, B]$ (complete bipartite graph), $A \cup B$ can be covered by the vertices of at most $2r-2$ monochromatic connected components. We reduce this conjecture to design-like conjectures, where the monochromatic components of the color classes are bicliques $[X, Y]$ with nonempty blocks X and Y. It can be also assumed that each color class covers $A \cup B$ (spanning), moreover, no X-blocks or Y-blocks properly contain each other (antichain property). We prove this reduced conjecture for $r \leq 5$.

We show that the width (the number of bicliques) in every color class of any spanning r-coloring is at most 2^{r-1} (and this is best possible). On the other hand there exist spanning r-colorings such that the width of every color class is $\Omega(r^{3/2})$.

We discuss the dual form of the conjecture which relates to transversals of intersecting and cross-intersecting r-partite hypergraphs.

1 Introduction, summary of results.

A special case of a conjecture generally attributed to Ryser (appeared in his student, Henderson's thesis, [8]) states that intersecting r-partite hypergraphs have a transversal of at most $r-1$ vertices (see Conjecture 4 in Section 6). This conjecture is open for $r \geq 6$. It is trivially true for $r = 2$, the cases $r = 3, 4$ are solved in [3] and in [2], and for the case $r = 5$, see [2] and [13]. The following equivalent formulation is from [3],[5]:

Conjecture 1. *In every r-coloring of the edges of a complete graph, the vertex set can be covered by the vertices of at most $r-1$ monochromatic connected components.*

Gyárfás and Lehel discovered a bipartite version of this conjecture [3], [9]. A complete bipartite graph G with non empty vertex classes X and Y is referred to as a *biclique* $[X, Y]$ in this paper, and X and Y will be called the *blocks* of this biclique. Given an edge coloring, a *monochromatic component* means a connected component of the subgraph of any given color. The number of components in a given color is called the *width of the color.*

Conjecture 2. *In every r-coloring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2r-2$ monochromatic components.*

First we show here that Conjecture 2, if true, is best possible. Let $G^* = [A, B]$ be a biclique with $|A| = r-1, |B| = r!$, and label the vertices of A with $\{1, 2, \ldots, r-1\}$ and those of B with the $(r-1)$-permutations of the elements of $\{1, 2, \ldots, r\}$. For $k \in A$ and $\pi = j_1 j_2 \ldots j_{r-1} \in B$, let the color of the edge $k\pi$ be j_k.

Since each vertex in B is incident with $r-1$ edges of distinct color, every monochromatic component of G^* is a star with $(r-1)!$ leaves centered at A. Furthermore, G^* has a vertex cover with $2r-2$ monochromatic components, just take the r monochromatic stars centered at vertex $r-1$, and add one edge from each vertex $k = 1, 2, \ldots, r-2$ of A.

Proposition 1. ([3]) *The vertex set of G^* cannot be covered with less than $2r-2$ monochromatic components.*

Proof. Let $\mathcal{C}$ be a cover of $V(G^*) = A \cup B$ by monochromatic stars centered in A. Let a_k denote the number of monochromatic stars of $\mathcal{C}$ on vertex $k \in A$. We may assume that $a_1 \leq a_2 \leq \dots \leq a_{r-1}$.

We show first that $a_i \geq i+1$ holds for some $1 \leq i \leq r-1$. Suppose on the contrary that $a_i \leq i$, for all i. Thus we can select a color $j_{r-1} \in \{1, \dots, r\}$ different from the a_{r-1} colors of all stars of $\mathcal{C}$ centered at $r-1$. Then we can select a new color $j_{r-2} \in \{1, \dots, r\} \setminus \{j_{r-1}\}$ different from the a_{r-2} colors of all stars of $\mathcal{C}$ centered at $r-2$, etc. Thus we end up by selecting $r-1$ distinct colors $j_1, \dots, j_{r-1}$. This is a contradiction since the $(r-1)$-permutation $j_1 j_2 \dots, j_{r-1} \in B$ is uncovered by $\mathcal{C}$.

Now let $a_i \geq i+1$, for some $1 \leq i \leq r-1$, then the number of stars in $\mathcal{C}$ is

$$\sum_{k=1}^{r-1} a_k = \sum_{k=1}^{i-1} a_k + \sum_{k=i}^{r-1} a_k \geq (i-1) + (i+1)(r-i).$$

Because

$$(i-1) + (i+1)(r-i) = -i^2 + ri + r - 1 \geq 2r - 2$$

holds for every $1 \leq i \leq r-1$, the proposition follows. □

It is worth noting that Conjecture 2 (similarly to Conjecture 1) becomes obviously true if the number of monochromatic components is just one larger than stated in the conjecture.

Proposition 2. ([3]) *In every r-coloring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2r-1$ monochromatic components.*

Proof. For an edge uv of the biclique G, consider the monochromatic component (double star) formed by the edges in the color of uv incident to u or v. In all other colors consider the (at most $r-1$) monochromatic stars centered at u and at v. This gives $2r-1$ monochromatic components covering the vertices of G. □

In Section 2 we show that Conjecture 2 can be reduced to design-like conjectures: one can assume that all colors span *bi-equivalence graphs*, i.e. graphs whose components are complete bipartite graphs. It is worth noting that a similar reduction is not known for Conjecture 1.

Our results in Sections 4 and 5 (Corollary 1, Theorems 5, 6) imply

Theorem 1. *Conjecture 2 holds for $r = 2, 3, 4, 5$.*

Suppose we have a partition of a biclique into bi-equivalence graphs. We call a pair u, v in one of the cliques of a biclique *equivalent* if u and v belong to the same block in every bi-equivalence graph. Since equivalent vertices do not change the number of components needed for a cover, the following result shows that for every fixed r one has to consider only finitely many colorings.

Theorem 2. *Suppose a biclique* $[A, B]$ *is partitioned into* r *bi-equivalence graphs and no two vertices of* A *are equivalent. Then* $\max\{|A|, |B|\} \le r!$ *and equality is possible.*

It is natural to ask how many monochromatic components (or bicliques) *of the same color* cover all vertices in r-colorings of bicliques, i.e. to bound the minimum width of the color classes. Such coverings are called *homogeneous* in Section 3. In the example G^* used in Proposition 1 the width of every color class is $(r-1)!+r-1$ (this property of G^* played a role in [6] where coverings by monochromatic cycles have been studied). However, for *spanning colorings*, where at least one edge is adjacent in each color to any vertex, the situation is different: using a deep result of Alon [1], we show

Theorem 3. *In a spanning* r*-coloring the width of every color class is at most* 2^{r-1} *and this is best possible.*

It is tempting to conjecture (in fact one of the authors did) that Conjecture 2 is true in a stronger form: *some* color class in every spanning r-coloring has width at most $2r - 2$. However,

Theorem 4. *There are spanning* r*-partitions of bicliques such that the width of every partition class is* $\Omega(r^{3/2})$.

Theorem 4 naturally suggests the following question.

Question 1. *Determine or estimate* $g(r)$*, the largest* m *such that there is a spanning partition of a biclique into* r *bi-equivalence graphs, all with width at least* m.

Very recently T. Terpai [12] improved the bound of Theorem 4 to $g(r) = \Omega(r^2)$.

In Section 6 we formulate the dual forms of Conjectures 1, 2 and show their relation to transversals of intersecting and cross-intersecting r-uniform hypergraphs.

2 Equivalent conjectures, notations.

Here we prove some equivalent forms of Conjecture 2 leading to a design-like conjecture (Conjecture 3). In this spirit an r-coloring will be also called a partition of the edge set into r subgraphs.

A. *If a biclique is partitioned into* r *bi-equivalence graphs, then its vertex set can be covered by at most* $2r - 2$ *biclique components.*

Since the bi-equivalence graphs in claim A can be color classes of an r-coloring, validity of Conjecture 2 implies that claim A is also true.

On the other hand, suppose we have an r-coloring of a biclique $G = [X, Y]$ such that some monochromatic component C, say in color 1, is not a biclique. Let $x \in X, y \in Y$ be non-adjacent vertices in C, w.l.o.g. xy has color 2. Observe that the $2(r-2)$ monochromatic stars in colors $3, \dots, r$ centered at x and at y, plus the component C, and the component in color 2 containing xy cover $V(G)$, leading to a cover with at most $2r-2$ monochromatic components. Thus Conjecture 2 follows from claim A.

Let us call a bi-equivalence graph partition $G_1,\dots,G_r$ of biclique G a *spanning partition* if each vertex $v \in V(G)$ is included in every $V(G_i)$, $i=1,\dots,r$. Notice that it is enough to prove claim A for spanning partitions. Indeed, assuming that $v \notin V(G_1)$ and $vw \in E(G_2)$, just take the at most $r-2$ bicliques from $G_3,\dots,G_r$ that contain v and add the at most r bicliques from $G_1,G_2,\dots,G_r$ that contain w, together they form a cover of all vertices of G with at most $2r-2$ bicliques. Thus we have the following equivalent form of claim A:

B. *If a biclique has a spanning partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2r-2$ biclique components.*

Let a biclique $[X,Y]$ be partitioned into the bi-equivalence graphs $G_1,G_2,\dots,G_r$. Then we will say that i is the color of the edges in G_i ($i=1,\dots,r$). Any connected component of G_i is a biclique, its vertex classes will called *blocks in color i*.

Denote by $B_i[u_1,\dots,u_k]$ the connected component of G_i which contains the vertices $u_1,\dots,u_k$, if they are in the same component of G_i, and in this case let $X_i[u_1,\dots,u_k] = X \cap V(B_i[u_1,\dots,u_k])$ and $Y_i[u_1,\dots,u_k] = Y \cap V(B_i[u_1,\dots,u_k])$ be the corresponding blocks. Otherwise set $B_i[u_1,\dots,u_k]=\emptyset$, $X_i[u_1,\dots,u_k]=Y_i[u_1,\dots,u_k]=\emptyset$.

Note that $B_i[u] \neq \emptyset$ for any $u \in V(G)$ in a spanning partition. In the sequel we will also use the fact that the blocks $X_i[u]$ and $X_i[v]$ are either disjoint or equal for any color $i \in \{1,2,\dots,r\}$ and any vertices $u,v \in V(G)$.

Let us call a spanning bi-equivalence graph partition $G_1,\dots,G_r$ of biclique G an *antichain partition* if no two blocks properly contain each other, that is if no colors $i,j \in \{1,\dots,r\}$ and no vertices $u,v \in V(G)$ exist such that $X_i[u] \subsetneq X_j[v]$ or $Y_i[u] \subsetneq Y_j[v]$.

If $v \in X$ and $|X_i[v]|=1$ (or $v \in Y$ and $|Y_i[v]|=1$) then we call vertex v a *singleton* block in color i. Note that if a coloring has the antichain property, then a singleton block in some color is a singleton in every color, in this case we just say that v is a singleton.

It turns out that it is enough to prove claim B for antichain partitions. Indeed, assume that in a spanning partition there are two blocks properly containing each other, that is $X_1[z] \subsetneq X_2[x]$, for some biclique components $B_1[z]$ and $B_2[x]$. Assume that $x \notin X_1[z]$, and let $y \in Y_1[z]$. The color of the edge xy is neither 1 nor 2, w.l.o.g. it is 3. Because $B_3[y]=B_3[x]$ and $X_1[y]=X_1[z] \subseteq X_2[x]$, the collection

$$\{B_i[x] \ : \ i \in \{1,2,\dots,r\}\} \cup \{B_i[y] \ : \ i \in \{1,2,\dots,r\} \setminus \{1,3\}\}$$

is a cover with at most $2r-2$ monochromatic components. Thus we obtain the following equivalent form of Conjecture 2.

Conjecture 3. *If a biclique has an antichain partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2r-2$ biclique components.*

Finally we note an important reduction used extensively in the proofs later. We recall that a pair $u, v \in A$ or $u, v \in B$ *equivalent* if in every bi-equivalence graph of the bi-equivalence graph partition of the biclique G, u and v belong to the same block. We may assume w.l.o.g. that there is no pair of equivalent vertices, and in this case we say that *the coloring is reduced*. Indeed, if there were two vertices u, v such that $uv \notin E(G)$ and for every $w \in V(G)$ with $uw, vw \in E(G)$, the edges uw and vw have the same color, then v could be added to any monochromatic component of $G - \{v\}$ containing u. Hence if Conjecture 3 holds for $G - \{v\}$ then it also holds for G.

Theorem 2. *Suppose a biclique $[A, B]$ has a partition into r bi-equivalence graphs and no two vertices of A are equivalent. Then* $\max\{|A|, |B|\} \leq r!$ *and equality is possible.*

Proof. It is easy to check that the partition of G^* into bi-equivalence graphs in Proposition 1 is a reduced one, hence the second statement follows.

To prove the first statement, the case $r = 1$ is obvious. Assuming it is true for some $r \geq 1$, suppose indirectly that $|A| \geq (r+1)! + 1$ in some partition into $r+1$ bi-equivalence graphs. Then for any fixed $v \in B$ there are $r! + 1$ edges of the same color from v, say in color $r+1$, to $Y \subset A$. Let X be the set of vertices in B that send edges in at least two different colors to Y. By the assumption $X \neq \emptyset$ and since color class $r+1$ is a bi-equivalence graph, $[X, Y]$ has no edge of color $r+1$. This means no two vertices of Y are equivalent in the induced r-partition on $[X, Y]$, and thus $|Y| > r!$ contradicts the inductive hypothesis. □

3 Homogeneous covering.

In 1998 Guantao Chen asked whether a stronger version of claim B can be true, i.e. whether $2r-2$ biclique components *of the same bi-equivalence graph* G_i, $1 \leq i \leq r$, can cover $[X, Y]$. Call such cover a *homogeneous cover*. Although this is not true in general (see Theorem 4 below), the question introduces interesting variants of the cover problem.

Given r, let $g(r)$ be the smallest m such that in every biclique B with a spanning partition into r bi-equivalence graphs $G_1, \dots, G_r$, there is a partition class G_i with width at most m. We shall prove that $g(r)$ exists, in a stronger form: for every r, there is a smallest $m = h(r)$ such that in every spanning partition of a biclique into r bi-equivalence graphs, the width of *every partition class* is at most m.

Theorem 3. $h(r) = 2^{r-1}$.

Proof. To see that $h(r) \geq 2^{r-1}$ consider the following easy recursive construction to partition a biclique into r bi-equivalence graphs such that the maximum width is 2^{r-1}. The case $r = 1$ is obvious. Given such a spanning partition of $B = K_{n,n}$ into r bi-equivalence classes, take two vertex disjoint copies of B and place two bicliques crosswise as the $r+1$-th partition. This way a spanning partition of $K_{2n,2n}$ is obtained into $r+1$ bi-equivalence graphs and the width of every partition class is doubled - apart from the $(r+1)$-th class which has width two.

To prove the other direction, $h(r) \leq 2^{r-1}$, we need some definitions. An equivalence graph is a graph whose components are complete graphs. Let $eq(G)$ denote the minimum number of spanning equivalence graphs needed to cover the edge set of a graph G. Similarly, for any bipartite graph G, let $eqbi(G)$ denote the minimum number of spanning bi-equivalence graphs needed to cover the edges of G. Let G^+ denote the graph obtained from the bipartite graph G by adding to $E(G)$ all pairs inside the partite classes of G. Let $K_{t,t}^- = K_{t,t} - tK_2$, i.e. $K_{t,t}^-$ is a balanced biclique from which a perfect matching is removed. We need the next two straightforward propositions.

Proposition 3. *For any bipartite graph G, $eqbi(G) \geq eq(G^+) - 1$.*

Proof. Consider an optimal cover of $E(G)$ with $eqbi(G)$ spanning bi-equivalence graphs and turn them into spanning equivalence graphs by adding all missing edges to all biclique components. These plus one more spanning equivalence graph formed by the two vertex classes of G cover all edges of G^+ thus $eqbi(G) + 1$ is an upper bound of $eq(G^+)$. □

Proposition 4. *If B is a biclique and $G = B - E(H)$, where H is a spanning bi-equivalence subgraph of B with $t \geq 2$ components, then $eqbi(G) = eqbi(K_{t,t}^-)$.*

Proof. Suppose X_i, Y_i are the bicliques of H, $i = 1, 2, \ldots, t$ and $x_i y_i$ are the removed edges of $K_{t,t}$.

If $\{H_l : 1 \leq l \leq s\}$ is a spanning partition of $K_{t,t}^-$ into bi-equivalence graphs, define G_l by adding all edges of all bipartite graphs $[X_i, Y_j]$ whenever $x_i y_j$ is an edge of biclique of H_l. This defines $\{G_l : 1 \leq l \leq s\}$ as a spanning partition of G into bi-equivalence graphs showing that $eqbi(G) \leq s$.

To see the reverse inequality, consider an arbitrary cover of G by spanning bi-equivalence graphs $G_1, \ldots, G_k$. Let T be the subset of $2t$ vertices of $V(G)$ containing one vertex from each partite class of each bipartite component of H. For any $1 \leq l \leq k$, define H_l as the induced subgraph of G_l on T. Then $\{H_l : 1 \leq l \leq k\}$ is a spanning partition of $K_{t,t}^-$ into bi-equivalence graphs showing that $k \geq eqbi(K_{t,t}^-)$. □

The main tool is the following result of Alon [1].

Theorem ([1]). *Suppose that the maximum degree of the complement of a graph G is d and $|V(G)| = n$. Then $eq(G) \geq \log_2 n - \log_2 d$.*

Suppose indirectly that B is a biclique with a spanning partition into bi-equivalence graphs $G_1, \ldots, G_r$ such that some of them, say G_1 has width $t > 2^{r-1}$. Let $G = B - G_1$. Using Propositions 3, 4 and Alon's theorem, we obtain that

$$eqbi(G) = eqbi(K_{t,t}^-) \geq eq((K_{t,t}^-)^+) - 1 \geq \log_2(2t) - \log_2 1 - 1 > \log_2(2^r) - 1 = r - 1$$

which is a contradiction since the $r-1$ bi-equivalence graphs $G_2, \ldots, G_r$ partition $G = B - G_1$. Consequently $t \leq 2^{r-1}$, and $h(r) = 2^{r-1}$follows. This concludes the proof of Theorem 3. □

Next we prove Theorem 4, a lower bound for $g(r)$.

Theorem 4. *There are spanning r-partitions of bicliques such that the width of every partition class is $\Omega(r^{3/2})$.*

Proof. Let $s \geq 3$ be an integer, set $r = (s-2)s$ and $p = \binom{s-1}{2}$. We shall construct a spanning r-partition of the biclique $K_{sp,sp}$ into bi-equivalence graphs such that each class will be the disjoint union of one copy of the biclique $K_{p,p}$ and $s-1$ copies of the matching pK_2. Notice that each of those r classes has width $p(s-1)+1 \geq cr^{3/2}$, with constant c.

The construction is as follows. Let us color the edges of a Hamiltonian cycle of $K_{s,s}$ red, and all the other edges of $K_{s,s}$ blue. Each of the $s^2 - 2s = r$ blue edges can be uniquely extended with $s-1$ red edges into a 1-factor of $K_{s,s}$. Therefore, each red edge belongs to the same number, $r(s-1)/2s = p$ such 1-factors. Now we replace each vertex by a set of p elements, every blue edge with a copy of $K_{p,p}$, and every red edge with p pairwise disjoint copies of pK_2. □

The lower bound of Theorem 4 is recently improved by T. Terpai, [12]. In fact his construction is not only spanning but also an antichain partition. What we know about the functions g and h is $\Omega(r^2) = g(r) \leq h(r) = 2^{r-1}$, and it is a challenging question how they separate.

4 Bi-equivalence partitions for $r = 2, 3$ and 4.

In the present section we prove Conjecture 2 for the small cases in strongest possible form.

Proposition 5. *If a biclique $[X, Y]$ is partitioned into at most two bi-equivalence graphs, then each has at most two (non trivial) connected components.*

Proof. Assume on the contrary that $x_j y_j, j = 1, 2, 3$, are three edges from three distinct connected components of G_1, where $x_j \in X$ and $y_j \in Y$. Then the path (x_1, y_2, x_3, y_1) is in G_2, but the color of $x_1 y_1$ is not 2. Hence G_2 is not bi-equivalence graph, a contradiction. □

Proposition 6. *Let a biclique $[X, Y]$ be partitioned into three bi-equivalence graphs. If one of those has more than three non trivial components, then some of the other two is spanning and has two connected components.*

Proof. Assume on the contrary that $x_j y_j, j = 1, 2, 3, 4$, are four edges from four distinct connected components of G_1, where $x_j \in X$ and $y_j \in Y$.

The subgraph of the biclique on the vertex set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ contains a 6-cycle C whose edges are colored with 2 and 3. Since the color classes are bi-equivalence graphs, C has no monochromatic path of length more than two.

First assume that C has three edges of color 2 (the other three are colored with 3). W.l.o.g. we assume that $x_4 y_1, x_4 y_2 \in E(G_2)$. By the bi-equivalence property, we have $x_1 y_2, x_2 y_1 \in E(G_3)$. Since C has three edges in G_2, we may assume $y_1 x_3, y_2 x_3 \in E(G_2)$.

Observe that the edges x_1y_3, x_2y_3 of C are colored differently from the set $\{2,3\}$ hence the color of x_4y_3 is neither 2 nor 3, a contradiction.

Therefore C has four edges in one color and two edges in the other color. W.l.o.g. we assume that the colors follow each other along the cycle $C = (x_1, y_3, x_2, y_1, x_3, y_2, x_1)$ as $2, 2, 3, 2, 2, 3$. Then for every vertex $x \in X \setminus (X_1[x_1] \cup X_1[x_2] \cup X_1[x_3])$ we obtain that $xy_1, xy_2 \in E(G_2)$. Observe that this is also true for every $x \in X_1[x_3]$, since the $(2,3)$-coloring pattern along the 6-cycle $C' = (C - x_3) + x$ uniquely determines the color of the two edges at x.

In the same way one obtains that $X \setminus (X_1[x_1] \cup X_1[x_2])$ and $Y_1[y_1] \cup Y_1[y_2]$ induce a biclique in G_2, since, for $i = 1, 2$, any vertex $y \in Y_1[y_i]$ can replace y_i in the cycle C without altering the $(2,3)$-coloring pattern along the modified cycle. By symmetry of X and Y, we obtain that $Y \setminus (Y_1[y_1] \cup Y_1[y_2])$ and $X_1[x_1] \cup X_1[x_2]$ induce a biclique of G_2 as well.

Therefore G_2 is spanning and has two connected components. $\square$

The propositions above imply immediately

Corollary 1. *For $r = 2, 3$, in any spanning partition of a biclique into r bi-equivalence graphs some color class has width at most r.*

With the antichain assumption Corollary 1 extends for $r = 4$ as well:

Theorem 5. *If a biclique has an antichain partition into four bi-equivalence graphs then the width of some color class is at most 4.*

Proof. Let $G_i, i = 1, 2, 3, 4$, be the bi-equivalence graphs in a reduced antichain partition of a biclique $[X, Y]$.

Claim 1: if $|X_i[u]| \leq 2$ for every color i and vertex u, then G_1 has 4 components.
To see this let $y \in Y$ and set $U = \bigcup_{i=2}^{4} X_i[y]$. Let s be the number of components of G_1 that intersect U at a single vertex. If $x \in X_i[y]$, for some $i \in \{2, 3, 4\}$, and $B_1[x] \cap U = \{x\}$, then $X_1[x] = \{x\}$ and hence by the antichain property, $X_i[y] = \{x\}$ follows. Thus for the number of components of G_1 different from $B_1[y]$ we obtain $s + 2(3 - s)/2 = 3$, and the claim follows.

Due to Claim 1 we may assume that there are three distinct vertices, $x_1, x_2, x_3 \in X$ in some block of G_1. Let

$$Y(c_1, c_2, c_3) = \{y \in Y \mid yx_i \text{ is colored with } c_i, i = 1, 2, 3\}.$$

The three-tuple (c_1, c_2, c_3) will be called the type of the subset $Y(c_1, c_2, c_3)$. In terms of this notation $Y(1, 1, 1) \neq \emptyset$. When the wildcard character $*$ is used for a color, then the color of the corresponding edge between $\{x_1, x_2, x_3\}$ and the set of that type is undetermined (e.g. $Y(3, 3, 4) \subseteq Y(3, *, 4)$ is true).

In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next claim:

Claim 2: If a, b are distinct colors, then at least one of the sets $Y(a, a, *)$ and $Y(a, b, *)$ must be empty. Indeed, if $y_1 \in Y(a, a, *)$ and $y_2 \in Y(a, b, *)$, then (y_2, x_1, y_1, x_2) is a path belonging to some biclique of G_a, hence the edge $x_2 y_2$ must have color a, and not b.

Using that x_1, x_2 are not equivalent we obtain the following

Claim 3: If $Y(2, 2, *) \neq \emptyset$ then $Y(3, 4, *)$ and $Y(4, 3, *)$ are not empty.

Claim 4: $Y(i, i, i) = \emptyset$ for every i in $\{2, 3, 4\}$. Assume on the contrary that $Y(2, 2, 2) \neq \emptyset$. Because x_1 and x_2 are not equivalent, we have $Y(3, 4, *) \neq \emptyset$, $Y(4, 3, *) \neq \emptyset$, and therefore, $Y(3, 3, *) = \emptyset$, $Y(4, 4, *) = \emptyset$. Moreover, this must hold for any pair x_i, x_j, $1 \leq i < j \leq 3$, which is impossible (by the pigeon hole principle).

Claim 5: At least one of $Y(2, 2, 3)$ and $Y(2, 2, 4)$ is empty. To see this, assume $Y(2, 2, 3) \neq \emptyset$ and $Y(2, 2, 4) \neq \emptyset$. By the previous claims we have

$$Y = Y(1, 1, 1) \cup Y(2, 2, 3) \cup Y(2, 2, 4) \cup Y(3, 4, 2) \cup Y(4, 3, 2),$$

where none of these types are empty. In particular $Y(*, *, 3) \cup Y(*, *, 4) \subseteq Y(2, *, *)$, violating the antichain property.

Now w.l.o.g. assume that either $Y(2, 2, 3) \neq \emptyset$ or in any (nonempty) type $Y(a, b, c)$ the elements a, b, and c are distinct, apart $Y(1, 1, 1)$. In both cases every (nonempty) type in $Y \setminus Y(1, 1, 1)$ has a color 3. Then the components $B_3[x_i], i = 1, 2, 3$, form a cover provided $Y_3[z] \cap (Y \setminus Y(1, 1, 1)) \neq \emptyset$, for all $z \in X$. If some z does not satisfy this, then by the antichain property, $Y(1, 1, 1) = Y_3[z]$, and $B_3[x_i]$, $i = 1, 2, 3$, and $B_3[z]$ together form a cover. □

5 Bi-equivalence partitions for $r = 5$.

In this section we shall verify Conjecture 3, for $r = 5$, in a stronger form. Actually we will show that under the appropriate conditions there is a cover with at most $2r - 2 = 8$ monochromatic components in the same color, or equivalently, one of the bi-equivalence graphs of the partition has width at most 8.

Theorem 6. *If a biclique has an antichain partition into five bi-equivalence graphs, then the width of some color class is at most* 8.

Let $G_i, i = 1, 2, 3, 4, 5$, be the bi-equivalence graphs in a reduced antichain partition of the biclique $[X, Y]$. For the proof we need two technical lemmas.

Lemma 1. *If each* $G_i, i = 1, \ldots, 5$, *has width at least* 6, *then* $[X, Y]$ *contains at most two singletons in each vertex class.*

Proof. Suppose on the contrary that one class has three singletons, say $x_1, x_2, x_3 \in X$ with $|X_i[x_j]| = 1$, for every $1 \leq i \leq 5$, and $1 \leq j \leq 3$. Then taking any $y \in Y_1[x_1]$, we may assume that $yx_2 \in E(G_2)$ and $yx_3 \in E(G_3)$. In particular, we obtain that $X = \{x_1, x_2, x_3\} \cup X_4[y] \cup X_5[y]$.

For any $z \in X_4[y]$, we have $X_5[z] \cap X_5[y] = \emptyset$, hence by the antichain property, $X_5[z] = X_4[y]$. Therefore G_5 has five components: $B_5[x_1]$, $B_5[x_2]$, $B_5[x_3]$, $B_5[z]$, $B_5[y]$, a contradiction. □

Lemma 2. *Let each $G_i, i = 1, \ldots, 5$, have width at least 9. If $[X, Y]$ contains at most two singletons in both of its vertex classes, then there is a color i and a vertex u for which $|X_i[u]| \geq 9$ or $|Y_i[u]| \geq 9$.*

Proof. Assume that for every color i and vertex u we have $|X_i[u]| \leq t$ and $|Y_i[u]| \leq t$. Let G_1 be the graph with the maximum number of edges among $G_i, i = 1, \ldots, 5$. The trivial inequality $|E(G)| \leq 5|E(G_1)|$ will give us a first lower bound on t.

For a vertex $u \in X$ we have $Y = Y_1[u] \cup Y_2[u] \cup Y_3[u] \cup Y_4[u] \cup Y_5[u]$. As $|Y_i[u]| \leq t$ we get $|Y| \leq 5t$. Similarly it follows that $|X| \leq 5t$. Since G contains at most two singletons, and the width of G_1 is at least 9 we have $5t \geq |Y| \geq 2 \cdot 1 + 7 \cdot 2 = 16$, therefore $t \geq 4$.

Let $\underline{x}$ and $\underline{y}$ be vectors which contain the sizes of the components of G_1 in X and in Y, respectively. Our assumptions on G_1 mean that the length of $\underline{x}$ and $\underline{y}$ is at least 9, they have at most two elements equal to 1, and all their elements are at most t. Using this notation $|E(G_1)| = \underline{x} \cdot \underline{y}$, and $|E(G)| = |X||Y| = (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1})$, where $\underline{1}$ is the constant 1 vector with appropriate length. We are going to investigate $\text{diff}(\underline{x}, \underline{y}) = |E(G)| - 5|E(G_1)| = (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1}) - 5(\underline{x} \cdot \underline{y})$, and determine its minimum over all possible values of $\underline{x}$ and $\underline{y}$. If this function is positive for some t, then there is no partition of G into graphs with the above conditions for the given value of t.

In the first steps we minimize $\text{diff}(\underline{x}, \underline{y})$, for any fixed $|X|$ and $|Y|$, that is we maximize $|E(G_1)| = \underline{x} \cdot \underline{y}$.

Step 1: We may assume that the length of $\underline{x}$ is equal to 9, and so the length of y is also 9. Otherwise we could join two components of G_1 and increase the number of edges. So we have $\underline{x} = (x_1, \ldots, x_9)$ and $\underline{y} = (y_1, \ldots, y_9)$.

Step 2: We can reorder the components of G_1 such that $\underline{y}$ is ordered non-increasingly. After that we may assume that the elements of $\underline{x}$ are also ordered non-increasingly. Otherwise we could swap two elements with $x_i < x_j$ for $1 \leq i < j \leq 9$ and this operation would not decrease the value of $\underline{x} \cdot \underline{y}$. (The increment is $(x_j - x_i)(y_i - y_j) \geq 0$.) Hence $y_1 \geq y_2 \geq \cdots \geq y_9$ and $x_1 \geq x_2 \geq \cdots \geq x_9$.

Step 3: For $j > i$, the operation of increasing x_i and decreasing x_j by the same constant c increases $|E(G_1)| = \underline{x} \cdot \underline{y}$ with $c(y_i - y_j) \geq 0$.
By repeated use of this operation (observing the condition that each element of $\underline{x}$ and $\underline{y}$ is at most t, and these vectors contain at most two elements equal to 1) we obtain that

$x_1 = \cdots = x_p = t$, $t > x_{p+1} \geq 2$, $x_{p+2} = \cdots = x_7 = 2$, $x_8 = x_9 = 1$ and similarly $y_1 = \cdots = y_q = t$, $t > y_{q+1} \geq 2$, $y_{q+2} = \cdots = y_7 = 2$, $y_8 = y_9 = 1$. From $|X| \leq 5t$ it follows that $p < 5$, and similarly we get $q < 5$.

Thus for a given $|X|$ and $|Y|$, the maximum value $|E(G_1)| = \underline{x} \cdot \underline{y}$ is determined by the vectors $\underline{x}, \underline{y}$ standardized as above. In the next steps we minimize $\text{diff}(\underline{x}, \underline{y})$ by changing $|X|$ and $|Y|$.

Step 4: If $x_{p+1} \neq 2$ then let $\underline{x}^-$ and $\underline{x}^+$ be vectors almost the same as $\underline{x}$, but at the $(p+1)$-th position they have $x_{p+1} - 1 \geq 2$ and $x_{p+1} + 1 \leq t$, respectively. We claim that $\text{diff}(\underline{x}^-, \underline{y})$ or $\text{diff}(\underline{x}^+, \underline{y})$ is not greater than $\text{diff}(\underline{x}, \underline{y})$. Indeed, $\text{diff}(\underline{x}, \underline{y}) - \text{diff}(\underline{x}^-, \underline{y}) = \text{diff}(\underline{x}^+, \underline{y}) - \text{diff}(\underline{x}, \underline{y}) = |Y| - 5y_{p+1}$ which means that $\text{diff}(\underline{x}, \underline{y})$ is a middle element of an arithmetic progression between $\text{diff}(\underline{x}^-, \underline{y})$ and $\text{diff}(\underline{x}^+, \underline{y})$. Thus we may assume that $x_{p+1} = 2$ and similarly $y_{q+1} = 2$. Furthermore we assume that $q = p + r$, where $r \geq 0$.

Step 5: Now we can express $\text{diff}(\underline{x}, \underline{y})$ as a function of p and r in the following way:

$$\begin{aligned} \text{diff}(\underline{x}, \underline{y}) &= (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1}) - 5(\underline{x} \cdot \underline{y}) \\ &= (tp + 2(7-p) + 2)(t(p+r) + 2(7-p-r) + 2) \\ &\quad -5(t^2 p + 2tr + 4(7-p-r) + 2), \end{aligned}$$

where the coefficient of r is $p(t-2)^2 + 6(t-2) > 0$, as $t \geq 4$. Therefore $\text{diff}(\underline{x}, \underline{y})$ is minimal if $r = 0$, that is $p = q$, and so $\underline{x} = \underline{y}$. In this case $\text{diff}(\underline{x}, \underline{x}) = p^2(t^2 - 4t + 4) + p(-5t^2 + 32t - 44) + 106$, which has extremum if $\frac{\mathrm{d}}{\mathrm{d}p}\text{diff}(\underline{x}, \underline{x}) = 0$ which gives $p = \frac{5t^2 - 32t + 44}{2(t^2 - 4t + 4)}$. (This extremum is a minimum since $\frac{\mathrm{d}^2}{\mathrm{d}p^2}\text{diff}(\underline{x}, \underline{x}) = 2(t^2 - 4t + 4) = 2(t-2)^2 > 0$, because $t \geq 4$.)

From the above formula we get $p = 1.5$, for $t = 8$, which gives that the minimum value of $\text{diff}(\underline{x}, \underline{y})$ for any $\underline{x}$, $\underline{y}$ is at least $25 > 0$. (Actually the minimum is 34 which is taken on the integer values $p = 1$ and $p = 2$.) Thus $|E(G)| \leq 5|E(G_1)|$ cannot hold for $t = 8$, which completes the proof. □

Proof of Theorem 6. Applying Lemmas 1 and 2, it follows that there is a block containing at least nine distinct vertices, say $x_i \in X_1[x_1]$, for every $i = 1, 2, \ldots, 9$. Similarly to the proof of Theorem 5, for a sequence of given colors $c_1, \ldots, c_9$, let

$$Y(c_1, \ldots, c_9) = \{y \in Y \mid yx_i \text{ is colored with } c_i, i = 1, \ldots, 9\}.$$

The nine-tuple $(c_1, \ldots, c_9)$ will be called the type of the subset $Y(c_1, \ldots, c_9) \subseteq Y$. In terms of this notation Lemmas 1 and 2 imply that $Y(1, \ldots, 1) \neq \emptyset$. Again, when the wildcard character $*$ is used for the i-th color position in a type, then the color of the corresponding edges to x_i are undetermined.

In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next rule.

Type rule. If a, b are distinct colors, then at least one of the sets $Y(a, a, *, \dots, *)$ and $Y(a, b, *, \dots, *)$ must be empty.

Indeed, if $y_1 \in Y(a, a, *, \dots, *)$ and $y_2 \in Y(a, b, *, \dots, *)$, then (y_2, x_1, y_1, x_2) is a path belonging to G_a, hence the edge $x_2 y_2$ must have color a, and not b.

Notice that the Type rule remains valid when permuting colors and/or when relabelling the vertices $x_1, x_2, \dots, x_9$, that is when the colors in the types are moved to different positions. Thus, for instance, types $(*, 5, *, \dots, *, 3)$ and $(*, 3, *, \dots, *, 3)$ cannot coexist.

We will need a simple corollary of the antichain property as follows:

Starring rule. If $Y_c[w] \subseteq Y(c_1, \dots, c_9)$, for some $w \in X$, then equality must hold because $Y(c_1, \dots, c_9) \subseteq Y(c_1, *, \dots, *) = Y_{c_1}[x_1]$, in that case we say that w "stars" the set $Y(c_1, \dots, c_9)$ in color c.

In the sequel when we write "w.l.o.g. we assume", we mean: "by appropriately permuting the colors and relabelling $x_1, x_2, \dots, x_9$ we may assume".

We shall proceed with investigating the partition of $Y' = Y \setminus Y(1, \dots, 1)$ into different types. Note that if $Y(c_1, \dots, c_9) \subseteq Y'$, then we have $c_i \neq 1$, for every $i = 1, \dots, 9$.

Distinguishing rule 1. If $Y(2, 2, *, \dots, *) \neq \emptyset$ and $Y(3, 3, *, \dots, *) \neq \emptyset$, then

$$Y(4, 4, *, \dots, *) \cup Y(5, 5, *, \dots, *) = \emptyset,$$

furthermore,

$$Y(4, 5, *, \dots, *) \neq \emptyset, \quad Y(5, 4, *, \dots, *) \neq \emptyset.$$

To see this recall that no equivalent vertices exist in the coloring, in particular x_1, x_2 must be distinguished by the components in colors 4 and 5. If $Y(4, 4, *, \dots, *) \neq \emptyset$, then by the Type rule, $B_i[x_1] = B_i[x_2]$ for every $i = 1, 2, 3, 4$, implying $B_5[x_1] = B_5[x_2]$, hence x_1, x_2 would be equivalent. An immediate corollary of Distinguishing rule 1 is stated for convenience as follows.

Distinguishing rule 2. At least one of $Y(2, 2, 2, *, \dots, *)$ and $Y(3, 3, 3, *, \dots, *)$ must be empty.

Returning to the proof let $Y(c_1, \dots, c_9) \subseteq Y'$. Since $c_i \in \{2, 3, 4, 5\}$, some color must repeat at least three times. We shall consider the following three cases:

1) there is a (nonempty) type in Y' such that a color repeats more than four times;

2) no (nonempty) type in Y' repeats a color more than four times, and there is a (nonempty) type repeating a color four times;

3) no (nonempty) type in Y' repeats a color more than three times.

Case 1: there is a (nonempty) type in Y' such that a color repeats more than four times, say $Y(2,2,2,2,2,*,\ldots,*) \neq \emptyset$.

Observe that color 2 cannot repeat seven times. Indeed, in every (nonempty) type in Y' different from $(2,2,2,2,2,2,2,*,*)$ color 2 is not used on the first seven positions, by the Type rule. Hence one color among $3, 4$, and 5 must repeat at least three times contradicting Distinguishing rule 2. Thus we may assume that $Y(2,2,2,2,2,*,c_7,*,*) \neq \emptyset$, where $c_7 \neq 2$.

A similar pigeon hole argument shows that in every (nonempty) type in Y' different from $(2,2,2,2,2,*,*,*)$ each of the three colors $3,4,5$ must be used on the first five positions, otherwise Distinguishing rule 2 is violated. Thus w.l.o.g. we assume that $c_7 = 3$.

Observe that by the Type rule, $Y_3[x_7] \subseteq Y(2,2,2,2,2,*,\ldots,*)$, thus by the Starring rule, $Y_3[x_7] = Y(2,2,2,2,2,*,\ldots,*)$ follows. Then we obtain that

$$Y' = (\cup\{Y_3[x_i] \mid 1 \leq i \leq 5\}) \cup Y_3[x_7].$$

If the six connected components $B_3[x_i], 1 \leq i \leq 5$ and $B_3[x_7]$ do not cover X, then there is an uncovered vertex $w \in X$ which stars $Y(1,\ldots,1)$ in color 3, by the Starring rule. In this case $B_3[x_i], 1 \leq i \leq 5$, $B_3[x_7]$, and $B_3[w]$ cover Y (thus the whole vertex set of G).

Consequently, in either case G_3 has width at most 7.

Case 2: no (nonempty) type in Y' repeats a color more than four times, and there is a (nonempty) type repeating a color four times, say $Y(2,2,2,2,c_5,\ldots,c_9) \neq \emptyset$, where $c_5,\ldots,c_9 \neq 2$. We also know that among the five colors, $c_5,\ldots,c_9$, there are two distinct colors, w.l.o.g. we assume that $c_5 = 3$ and $c_6 = 4$.

Assume now that in every (nonempty) type in Y' different from $(2,2,2,2,*,\ldots,*)$ color 3 is used somewhere on the first four positions. Then a similar argument that we used in Case 1 shows that the width of G_3 is at most 6. By the same reason repeated for color 4, it remains to consider the situation when, for each color 3 and 4, there is a (nonempty) type in Y' different from $(2,2,2,2,*,\ldots,*)$ missing 3 and 4 on the first four positions, respectively.

Since a color cannot repeat three times on the first four positions, we have that $Y(4,4,5,5,*,\ldots,*) \neq \emptyset$, moreover $Y(a,b,c,d,*,\ldots,*) \neq \emptyset$, where among a,b,c,d both colors 3 and 5 repeat twice. By the Type rule, either $a = b = 5, c = d = 3$ or $c = d = 5, a = b = 3$. In each case Distinguishing rule 1 is violated.

Case 3: no (nonempty) type in Y' repeats a color more than three times.

Then by the pigeon hole principle, each (nonempty) type in Y' has a color repeated three times. Furthermore, if a type uses just three colors, then each of its three colors is repeated exactly three times.

Let $Y(c,c,c,*,\ldots,*) \neq \emptyset$, for some $c = 2,3,4$, or 5. If each (nonempty) type uses color c at some position, then either the connected components $B_c[x_i], 3 \leq i \leq 9$ cover X, or some $w \in X$ stars $Y(1,\ldots,1)$ in color c, hence $B_c[x_i], 3 \leq i \leq 9$ and $B_c[w]$ cover Y (thus the whole vertex set of G). In each situation G_c has width at most 8. We claim that this must happen for some c.

Assume that color 2 repeats three times in some (nonempty) type, and some other (nonempty) type misses color 2. W.l.o.g. let $T_2 = (3,3,3,4,4,4,5,5,5)$ be a (nonempty) type. By repeating the same idea, we see that, for every $c = 3, 4, 5$, some (nonempty) type T_c misses c.

Thus T_3 has three triplets in colors $2, 4, 5$ at some positions. By Distinguishing rule 2 and the Type rule the last three positions of T_3 cannot be $5, 5, 5$. W.l.o.g. assume that $T_3 = (5, 5, *, 5, *, \ldots, *)$. Then again, by Distinguishing rule 2 and the Type rule, it follows that $T_3 = (5, 5, 4, 5, 2, 2, 4, 4, 2)$.

Finally, for the possible positions of the three 5's of T_4 with respect to T_2 and T_3, we conclude as before that $T_4 = (*, *, 5, *, 5, 5, *, *, *)$. This contradicts Distinguishing rule 1 on positions 5 and 6 and completes the proof of Theorem 6. □

6 The dual form, transversals of r-partite intersecting hypergraphs.

Conjectures 1 and 2 can be translated into dual forms as conjectures about transversals of r-partite r-uniform intersecting hypergraphs. To do that, one should consider the r partitions defined by the monochromatic connected components of an r-colored complete or complete bipartite graph as hyperedges over the vertex set and consider the dual of this hypergraph. This approach already turned out to be very useful, for example results of Füredi established in [4] can be applied. A survey on the subject is [7].

An r-uniform hypergraph H is defined by a finite set $V(H)$ called the vertex set of H, and by a set $E(H)$ of r-sets of $V(H)$ called edges of H. An r-uniform hypergraph H is called r-partite if there is a partition $V(H) = V_1 \cup \cdots \cup V_r$ such that $|e \cap V_i| = 1$, for all $i = 1, \ldots, r$ and $e \in E(H)$. A hypergraph H is called *intersecting* if $e \cap f \neq \emptyset$ for any $e, f \in E(H)$. A set $T \subseteq V(H)$ is called a transversal of H provided $e \cap T \neq \emptyset$, for all $e \in E(H)$; the minimum cardinality of a transversal of H is the transversal number of H denoted by $\tau(H)$.

The dual of Conjecture 1 is Ryser's conjecture for intersecting hypergraphs in its usual form as follows:

Conjecture 4. *If $\mathcal{H}$ is an intersecting r-partite hypergraph then $\tau(\mathcal{H}) \leq r - 1$.*

There are infinitely many examples of intersecting r-partite hypergraphs with transversal number equal to $r-1$. Take a finite projective plane of order q, then truncate it by removing one point and the incident $q+1$ lines. The remaining lines taken as edges define an intersecting $(q+1)$-partite hypergraph with transversal number equal to q. (Note that the truncated projective plane is the dual of an affine plane.) A related question, finding $f(r)$, the minimum number of edges among intersecting r-partite hypergraphs with transversal number at least $r-1$, was addressed in [11], where it was shown that $f(3) = 3$, $f(4) = 6$, and $f(5) = 9$.

Concerning our biclique cover conjectures, the dual of a spanning partition of a complete bipartite graph into r bi-equivalence graphs gives two r-partite hypergraphs, $\mathcal{H}_1, \mathcal{H}_2$ on the same vertex set such that for every $h_1 \in E(\mathcal{H}_1), h_2 \in E(\mathcal{H}_2)$, $|h_1 \cap h_2| = 1$ holds, moreover at each vertex there is at least one edge from both hypergraphs. We call such hypergraph pairs 1-cross intersecting. Then Conjecture 2 restated in claim B reads as follows:

Conjecture 5. *Let $\mathcal{H}_1, \mathcal{H}_2$ be a pair of 1-cross intersecting r-partite hypergraphs. Then $\tau(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2r - 2$.*

To illustrate the advantage of the dual formulation, here is a quick proof showing that $h(r)$ is bounded (although with a bound weaker than the one in Theorem 3).

Proposition 7. *Let $\mathcal{H}_1, \mathcal{H}_2$ be a pair of 1-cross intersecting r-partite hypergraphs. Then each partite class contains at most $\binom{2(r-1)}{r-1}$ vertices.*

Proof. Let $v_1, \ldots, v_p$ be the vertices of a partite class of $\mathcal{H}_1, \mathcal{H}_2$. For each v_i select $f_i^1 \in E(\mathcal{H}_1), f_i^2 \in E(\mathcal{H}_2)$ such that $v_i \in f_i^1 \cap f_i^2$, and set $g_i = f_i^1 \setminus \{v_i\}, h_i = f_i^2 \setminus \{v_i\}$. Then the pairs (g_i, h_i) form a cross-intersecting $r-1$-uniform family (in fact a very special one). It is well known (see Exercise 13.32 in [10]) that such hypergraphs have at most $\binom{2(r-1)}{r-1}$ edges. $\square$

References

[1] N. Alon, Covering graphs by the minimum number of equivalence relations, *Combinatorica* **6** (1986) 201–206.

[2] P. Duchet, Représentations, noyaux en théorie des graphes at hypergraphes, Thesis, Paris 1979.

[3] A. Gyárfás, Partition coverings and blocking sets in hypergraphs (in Hungarian) *Communications of the Computer and Automation Institute of the Hungarian Academy of Sciences* **71** (1977) 62 pp. MR 58, 5392.

[4] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981) 155–162.

[5] P. Erdős, A. Gyárfás, L. Pyber, Vertex coverings by monochromatic cycles and trees, *Journal of Combinatorial theory B* **51** (1991) 90–95.

[6] A. Gyárfás, M. Ruszinkó, G.N. Sárközy, E. Szemerédi, One-sided coverings of colored complete bipartite graphs, *In: Topics in Discrete Mathematics, Algorithms and Combinatorics, 26.* Springer, 2006, 133-154.

[7] A. Gyárfás, Large monochromatic components in edge colorings of graphs: a survey, In: Alexander Soifer (editor) Ramsey Theory: Yesterday, Today and Tomorrow. New York; London: Birkhauser Verlag, 2010. pp. 77–96.

[8] J. R. Henderson, Permutation Decomposition of (0-1)-Matrices and Decomposition Transversals, Ph.D. thesis, Caltech, 1971.

[9] J. Lehel, Ryser's conjecture for linear hypergraphs, manuscript, 1998.

[10] L. Lovász, Combinatorial Problems and Exercises, 2nd ed., AMS Chelsea Publishing, Providence, Rhode Island, 2007.

[11] T. Mansour, Ch. Song, R. Yuster, A comment on Ryser's conjecture for intersecting hypergraphs, *Graphs and Combinatorics* **25** (2009) 101–109.

[12] T. Terpai, Personal communication.

[13] Zs. Tuza, Some special cases of Ryser's conjecture, unpublished manuscript, 1978.