# Transitive Tournaments and Self-Complementary Graphs

**András Gyárfás**

*COMPUTER AND AUTOMATION RESEARCH INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, P.O. BOX 63, HUNGARY-1518.*

*E-mail: gyarfas@sztaki.hu*

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**Abstract:** A simple proof is given for a result of Sali and Simonyi on self-complementary graphs.  J Graph Theory 38: 111–112, 2001

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Motivated by Sperner capacities of digraphs, A. Sali and G. Simonyi [1] discovered an interesting property of self-complementary graphs. Here a proof of their result is given which is conceptually simpler than the original one.

**Theorem.** (Sali and Simonyi [1]) *For any self-complementary graph G on n vertices, the edges of the transitive tournament on n vertices can be partitioned into two isomorphic digraphs whose underlying graphs are isomorphic to G.*

***Proof.*** Assume that $\pi$ is a complementing permutation on the vertex set $[n] = \{1, 2, \ldots, n\}$ of $G$, i.e., for every $1 \leq x < y \leq n$, $xy \in E(G)$ implies $\pi(x)\pi(y) \notin E(G)$. The theorem is proved by defining a linear order $\alpha$ on $[n]$ such that $\pi$ *preserves* $\alpha$, i.e., for every $xy \in E(G)$ such that $x <_\alpha y$, it follows that $\pi(x) <_\alpha \pi(y)$. It is enough to define $\alpha$ on cycles of $\pi$ since if $\pi$ preserves a linear order on each cycle then $\pi$ preserves the sum of the linear orders. Let $C = \{1, 2, \ldots, k\}$ be a nontrivial cycle of $\pi$, i.e., $\pi(x) = x + 1 \pmod k$. We may assume that $12 \in E(G)$.

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We shall define $\alpha$ on $C$ by its initial segments $\{1\} = A_1 \subset \ldots \subset A_{k-1}$ so that $|A_i| = i$ and the elements of $A_i$ are (cyclically) consecutive on $C$. Assuming that $A_i$ is already defined for some $1 \leq i < k-1$, let $p$ and $s$ denote the predecessor and successor of the segment $A_i$ in the cyclic ordering of $C$. Since $i \leq k-2$, $p \neq s$. Then $A_{i+1}$ is defined by extending $A_i$ with $p$ if $ps \in E(G)$, or with $s$ if $ps \notin E(G)$.

To show that $\pi$ preserves $\alpha$, let $xy$ be an edge of $G$ with end points on $C$ such that $x <_\alpha y$. Select the smallest $i$ such that $x \in A_i$ and $y \notin A_i$. The definition of $i$ implies that $x$ is an end point of the segment $A_i$. This can happen in two ways.

**Case 1.** $A_i = \{x, \pi(x), \ldots\}$ in the cyclic order of $C$. From the choice of $A_i$, $x \notin A_{i-1}$ therefore $\pi(x) \in A_{i-1}, \pi(y) \notin A_{i-1}$, implying $\pi(x) <_\alpha \pi(y)$.

**Case 2.** $A_i = \{z, \ldots, x\}$ in the cyclic order of $C$. We claim that $\pi(y) = z$ is impossible. If $z = x$ then $i = 1$ and $12 = \pi(y)\pi(x) \notin E(G)$ contradicting the assumption $12 \in E(G)$. Otherwise $i \geq 2$ and $A_{i-1} = A_i \setminus \{x\}$ from the choice of $A_i$. However, from $\pi(y) = z$, $y$ is the predecessor of $A_{i-1}$. The successor of $A_{i-1}$ is $x$, $xy \in E(G)$, thus the definition of $A_i$ gives $A_i = A_{i-1} \cup \{y\}$, a contradiction proving the claim.

The claim implies that the segment $\pi(y) \ldots x$ in the cyclic order of $C$ properly contains $A_{i-1}$. Therefore, since $\pi(x)\pi(y) \notin E(G)$ and $\pi(x)$ is the successor of $x$, the chain $A_i \subset A_{i+1} \subset \cdots$ will absorb $\pi(x)$ before $\pi(y)$. Thus, for some $j \geq i$, $\pi(x) \in A_j$ and $\pi(y) \notin A_j$, i.e., $\pi(x) <_\alpha \pi(y)$. ■

## REFERENCE

[1] A. Sali and G. Simonyi, Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, Eur J of Combin 20 (1999), 93–99.