

Anti-Ramsey Colorings in Several Rounds

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A t -round χ -coloring is defined as a sequence ψ_1, \dots, ψ_t of t (not necessarily distinct) edge colorings of a complete graph, using at most χ colors in each of the colorings. For positive integers $k \leq n$ and t let $\chi^t(k, n)$ denote the minimum number χ of colors for which there exists a t -round χ -coloring of K_n such that all $\binom{k}{2}$ edges of each $K_k \subseteq K_n$ get different colors in at least one round. Generalizing a result of J. Körner and G. Simonyi (1995, *Studia Sci. Math. Hungar.* **30**, 95–103), it is shown in this paper that $\chi^t(3, n) = \Theta(n^{1/t})$. Two-round colorings for $k > 3$ are also investigated. Tight bounds are obtained for $\chi^2(k, n)$ for all values of k except for $k = 5$. We also study an inverted extremal function, $t(k, n)$, which is the minimum number of rounds needed to color the edges of K_n with the same $\binom{k}{2}$ colors such that all $\binom{k}{2}$ edges of each $K_k \subseteq K_n$ get different colors in at least one round. For $k = n/2$, $t(k, n)$ is shown to be exponentially large. Several related questions are investigated. The discussed problems relate to perfect hash functions. © 2001 Academic Press

1. ANTI-RAMSEY COLORINGS

The following problem has been considered by Simonovits and Sós [10]. Which colorings of the subgraphs isomorphic to a sample graph H must

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occur if the edges of K_n are colored by r colors? As they also remarked, this (general) setup does include Ramsey theory by choosing H to be monochromatic. On the other hand, anti-Ramsey type problems can be obtained if all edges of H are required to get different colors. Several results of this type were obtained by Erdős, Sós and Simonovits [4] and by Sós and Simonovits [10]: they obtained tight bounds for the maximum number of colors for which there exists a coloring of K_n without having a copy of some $H \subseteq K_n$ to be totally multicolored—i.e. at least one pair of edges of every subgraph H of K_n has to get the same color.

Here problems of this type will be investigated, but instead of one coloring, several colorings will be allowed. The requirement will be that for each H subgraph of K_n the anti-Ramsey criteria have to be satisfied in *at least one* of the colorings. In such case, clearly, there is a trade-off between the minimum number of *colors* and *colorings*. Actually, this problem has been also investigated—in a special case—by Sós. She asked for the minimum number of colorings of K_n with three colors such that each $K_3 \subseteq K_n$ in at least one the colorings is totally multicolored. Körner and Simonyi [8] showed that this minimum is between $\log_3 n$ and $\log_2 n$. In Section 3 this result will be generalized, showing the trade-off between the minimum number of colors and colorings if also more than three colors can be used. In Section 4 tight bounds for two-round colorings are given, and Section 5 contains results for the case when the anti-Ramsey criteria for large subgraphs of K_n have to hold. One can find the required notations and definitions in (next) Section 2. The obtained results and proof techniques relate to finite geometries and perfect hash functions. For more details on the latter topic and connections see [5, 7], and [8].

2. NOTATION

The complete graph on n vertices is denoted by K_n . A subgraph H of K_n is totally multicolored (colored rainbow) if all its edges have different colors. In the usual (anti)-Ramsey theory the extremal behavior of an edge-coloring is considered. Here, instead of one coloring, a sequence of χ -colorings ψ_1, \dots, ψ_t is considered and we want to ensure that every $K_k \subseteq K_n$ is totally multicolored in at least one coloring ψ_i ($1 \leq i \leq t$). According to this approach for positive integers $k \leq n$ and t let $\chi^t(k, n)$ denote the minimum number χ of colors such that there exists a sequence of length t of χ -colorings ψ_1, \dots, ψ_t of edges of K_n such that all $\binom{k}{2}$ edges of each $K_k \subseteq K_n$ get different colors in at least one coloring ψ_i . Such a sequence of colorings is a t -round coloring. Conversely, let $t(k, n)$ denote the minimum length of a sequence ψ_1, \dots, ψ_t of colorings of the edges of K_n with $\binom{k}{2}$ colors such that all $\binom{k}{2}$ edges of each $K_k \subseteq K_n$ get different colors in at least one coloring ψ_i .

The notations “ o ”, “ O ”, “ Θ ” and “ \sim ” are used in conventional sense, i.e., for sequences $f(n)$ and $g(n)$, $f(n) = o(g(n))$ ($f(n) \sim g(n)$) if $f(n)/g(n)$ tends to zero (one), respectively, as n tends to infinity, $f(n) = O(g(n))$ if $f(n) \leq cg(n)$ holds for some constant $c > 0$ and every n , and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $g(n) = O(f(n))$ hold. By \log the logarithm of base two will be denoted and \oplus stands for sum modulo two.

Let p be an arbitrary prime. Following the usual notation, $GF(q)$ is the finite (Galois) field of order $q = p^r$, and by $PG(m, q)$ ($AG(m, q)$) a projective (affine) geometry of dimension m over the finite field $GF(q)$ is denoted.

3. TRIANGLES

Coloring according to one-factors, it is clear that $\chi^1(3, n)$ is equal to $n - 1$ or n in case n is even or odd, respectively. Conversely, Körner and Simonyi [8] showed, that

THEOREM 3.1 (Körner, Simonyi). $\lceil \log(n - 1) / \log 3 \rceil \leq t(3, n) \leq \lceil \log n \rceil - 1$.

In the following theorem—unifying these two statements—the trade-off between the minimum number of colors and colorings is determined.

THEOREM 3.2. $\chi^t(3, n) = \Theta(n^{1/t})$, i.e., more precisely $(n - 1)^{1/t} \leq \chi^t(3, n) \leq 4n^{1/t} - 1$.

Proof. As it has been already observed by Körner and Simonyi [8], coloring the edges of K_n with χ colors, at least $(n - 1)/\chi$ edges adjacent to a given vertex x will get the same color. Iteratively, after the j th coloring still $(n - 1)/\chi^j$ edges adjacent to a given vertex x will get the same color in each of the first j colorings. On the other hand, every pair of edges having a common point have to get different colors at least in one of the colorings. So the number of colorings t has to be large enough to push the proportion $(n - 1)/\chi^t$ below one, from which the lower bound follows.

To prove the upper bound, assume that $2^{(k-1)+1} < n \leq 2^{k+1}$. In the following a sequence of $2^{k+1} - 1$ colorings will be presented, which rainbow colors each $K_3 \subseteq K_{2^{k+1}}$ in at least one of the colorings. By monotonicity, this will give the upper bound.

First we show the connection between the point colorings of $PG(m, 2)$ and edge colorings of K_n for $n \leq 2^{m+1}$. Take a hyper-plane $PG(m, 2)$ —say the set of points having the last coordinate zero—of $PG(m + 1, 2) = \{0, 1\}^{m+2} \setminus \{0\}^{m+2}$. The rest of the points will form an affine geometry $AG(m + 1, 2)$ (for more details on this topic see, e.g., the book of Hall [6]).

Assume that point colorings of $PG(m, 2)$ are given such that points of every line—each of them contains three points—are colored with different colors in at least one of the colorings. Now colorings of lines of $AG(m+1, 2)$ will be associated with given point colorings of $PG(m, 2)$ such that each triple of lines of $AG(m+1, 2)$ forming a triangle—i.e. any pair of them contain a common point—will get three different colors in at least one of the colorings. This will clearly lead to a rainbow coloring of K_n for $n \leq 2^{m+1}$, since lines of $AG(m+1, 2)$ consist of all pairs of its points and thus can be considered as edges of the complete graph $K_{2^{m+1}}$.

Take a line $\{x_i, x_j\}$ of $AG(m+1, 2)$. Since the last coordinate of both points is one, their $\text{mod}(2)$ sum is zero, and hence the third point x_{ij} of $PG(m+1, 2)$ being on the line determined by x_i and x_j is in the (fixed) hyper-plane $PG(m, 2)$. Now color the line $\{x_i, x_j\}$ of $AG(m+1, 2)$ in the l th coloring the same as $x_{ij} \in PG(m, 2)$ (in l th coloring) is colored. This is a rainbow coloring of lines of $AG(m+1, 2)$ by the following.

Take a triple x_i, x_j and x_k in $AG(m+1, 2)$. Denote by x_{ij}, x_{ik} , and x_{jk} the third points of the lines of $PG(m+1, 2)$ determined by the pairs $(x_i, x_j), (x_i, x_k), (x_j, x_k)$. As it was shown above, x_{ij}, x_{ik} , and x_{jk} are all in the (fixed) hyper-plane $PG(m, 2)$. Since at least in one of the given point colorings of $PG(m, 2)$ all points of the line $\{x_{ij}, x_{ik}, x_{jk}\}$ are colored differently, in the associated coloring the lines $\{x_i, x_j\}, \{x_i, x_k\}, \{x_j, x_k\}$ are colored differently, too. Hence a rainbow coloring of $K_{2^{m+1}}$ is obtained.

It remains to show that points of lines of $PG(m, 2)$ can be colored rainbow with the claimed number of colors and colorings. This follows from the next claim.

Claim 3.3. Points of $PG(tk, 2)$ can be colored with $2^{k+1} - 1$ colors using t colorings such that points of every line get different colors in at least one coloring.

The proof goes by induction on t . For $t=1$ this is trivially true, just color all $2^{k+1} - 1$ points of $PG(k, 2)$ with different colors.

Assume that for $1 \leq i \leq t-1$ the statement is true. Fix a subspace $S = PG((t-1)k - 1, 2)$ of $PG(tk, 2)$, say, taking all points with the last $k+1$ coordinates equal to zero. Partition the rest of the points of $PG(tk, 2) \setminus S$ into parts S_i ($i=1, \dots, 2^{k+1} - 1$) according to their last $k+1$ coordinates. Clearly, $S \cup S_i$ are subspaces of dimension $(t-1)k$. In the first coloring—using $2^{k+1} - 1$ colors—color every point of S_i with color i . Notice, that after this first coloring, every line containing no point from S is colored rainbow. Indeed, if $x_i \in S_i$ and $x_j \in S_j$ and $S_i \neq S_j$, then the third point on the line spanned by x_i and x_j is $x_i \oplus x_j \in S_k$, and $i \neq k \neq j$, since the last $k+1$ coordinates of $x_i \oplus x_j$ differ from the last ones of x_i and x_j , too. (Neither the last $k+1$ coordinates of x_i nor of x_j are zeros.) On the other hand, $x_i \oplus x_j \in S$ if and only if both x_i and x_j are from the same S_i (or S).

To color rainbow the rest of the triples—i.e. those containing points from S —we can use induction: triples of $S \cup S_i$ can be rainbow colored (by induction) with $2^{k+1} - 1$ colors and with $t - 1$ colorings simultaneously.

Putting this claim together with associated coloring of lines of $AG(m + 1, 2)$ gives the desired results.

Remark 3.4. By choosing the number of colors χ to be three, one gets the Theorem 3.1 of Körner and Simonyi [8] up to the correct order of magnitude.

It would be interesting to know the exact coefficient $1 \leq c \leq 2$ for which $\chi^t(3, n) = cn^{1/t} \pm o(n^{1/t})$. From the following construction for $t = 2$, one may get the impression that our upper bound is not tight.

LEMMA 3.5. *For arbitrary power of prime $q = p^r$, $q \leq \chi^2(3, q^2) \leq q + 1$.*

Proof. From Theorem 3.2, $\chi^2(3, q^2) \geq (q^2 - 1)^{1/2}$ follows, and since χ is integer, the lower bound is true.

To prove the upper bound partition the lines of the affine plane $AG(2, q)$ into $q + 1$ parallel classes, i.e. each class will contain q parallel lines. In the first coloring all $\binom{q}{2}$ edges of K_q are colored i if these q points are on a line in the i th parallel class. (More illustratively, all lines in a given parallel class will get the same color.) Since any pair of points determine a unique line and it is in a unique parallel class, this will determine a coloration of K_{q^2} with $q + 1$ colors.

Notice, that if three points are not collinear, the spanned K_3 is colored in the first coloring rainbow. To the contrary, assume that x_i, x_j and x_k are not collinear and the edges, say, (x_i, x_j) and (x_j, x_k) are colored the same. This means that lines of $AG(2, q)$ spanned by (x_i, x_j) and (x_j, x_k) are in the same color class. But then they are parallel, contradicting that x_j lies on both of them.

Therefore, in the second coloring it is enough to rainbow color collinear triples. In order to do this, partition the edges of each K_q induced by lines into one-factors, and color the i th one-factor of every factorization with (color) i simultaneously. Clearly, this will rainbow color the rest of triples. On the other hand, the number of one-factors in each factorization is $q - 1$ or q , for even or odd q , respectively. Therefore, in the second round at most q colors were used, while in the first round the number of colors used is $q + 1$, which finishes the proof. ■

Recall, that powers of primes are rather dense, so Lemma 3.5 and monotonicity and also Theorem 3.2 implies the following.

COROLLARY 3.6. *For $n \geq 3$, $(n - 1)^{1/2} \leq \chi^2(3, n) \leq n^{1/2}(1 + o(1))$.*

Also note, that the gap in the above inequality in Lemma 3.5 for powers of primes is at most one. The following lemma gives the impression that for powers of primes the construction of Lemma 3.5 is tight.

LEMMA 3.7. $\chi^2(3, 9) = 4$.

Proof. Assume that there are two colorings of K_9 each with 3 colors such that each triangle is rainbow colored. A straightforward case analysis will verify that each of the following must be true, or there will be some K_3 that will not be rainbow colored in either of two colorings.

1. Each vertex has degree at most 3 in any one color. Therefore, each vertex has color degree sequence precisely $(3, 3, 2)$ in each of the colorings.
2. There is no monochromatic C_5 in either of the colorings.
3. There is no monochromatic $K_{2,3}$ in either of the colorings.
4. There is no monochromatic K_4 in either of the colorings.

Note that (1) follows from the fact that if there is a monochromatic $K_{1,4}$ in the first coloring, then there will be a monochromatic $K_{1,2}$ in the second coloring using two of the edges of the $K_{1,4}$. To verify (2) observe that a monochromatic C_5 in the first coloring implies a unique (up to the order of the colors) coloring of the edges of the C_5 in the second coloring, and this includes the chords of the C_5 as well. Thus, in the second coloring none of the triangles using just one edge of the original C_5 are rainbow colored, and it is impossible to color all of the corresponding triangles rainbow in the first coloring. The same reasoning will verify (3) by using the monochromatic $K_{2,3}$ instead of the C_5 . Note that (4) follows from the fact that the graph spanned by the 5 vertices not in the monochromatic K_4 have degree at least 2 in the color of the K_4 , and thus must contain either a C_5 or a $K_{2,3}$.

There are $\binom{9}{3} = 84$ triangles in K_9 , and there are $9(\binom{3}{2} + \binom{3}{2} + \binom{2}{2})$ triples that are not rainbow colored in each coloring. However, the triples that come from monochromatic triangles are duplicated 3 times among these triples. If no color has 4 or more monochromatic triangles, then the number of triangles that are not rainbow is at least $63 - 2(3)(3) = 45$ in each of the two colorings. This implies that some triangle is not rainbow in either of the colorings. Thus, there are at least 4 triangles in some color in one of the colorings.

Let H be the monochromatic graph in one of the colors that has at least 4 triangles. The graph H contains no C_5 , $K_{2,3}$, or K_4 , and each vertex of H has degree either 2 or 3. Thus, any two triangles in H must be vertex disjoint or share an edge. These conditions can be used to show that H

consists of two vertex disjoint $(K_4 - e)$'s with a path of length 2 between a pair of degree 2 vertices in different $(K_4 - e)$'s, and possibly an edge between the remaining two vertices of degree 2. A case analysis will show that the monochromatic H cannot be extended to two 3-colorings of K_9 , with each triangle rainbow colored in one of the colors. This gives a contradiction. ■

It is worth mentioning, that by the Körner–Simonyi Theorem $\lceil \log(n-1)/\log 3 \rceil \leq t(3, n) \leq \lceil \log n \rceil - 1$, which gives $2 \leq t(3, 9) \leq 3$. The above lemma says that to color rainbow all subtriangles of K_9 in two rounds four colors are needed, from which $t(3, 9) = 3$ follows. This shows that for the smallest non-trivial case $n=9$ the upper bound of Körner–Simonyi theorem is tight.

4. COLORINGS IN TWO ROUNDS

The previous section ended with a construction for a rainbow coloring in two rounds for triangles. In this section we continue to investigate two-round rainbow colorings for bigger complete subgraphs of K_n .

THEOREM 4.1. *For infinitely many n , $((1 - 1/n)^{1/2}/\sqrt{2})n \leq \chi^2(4, n) \leq n$.*

Proof. The lower bound does hold for all positive integers n . Coloring the edges of K_n with k colors, after the first coloring, at least $\binom{n}{2}/k$ edges will get the same color. Observe, that any pair of edges is contained in some $K_4 \subseteq K_n$. Hence, all of those $\binom{n}{2}/k$ edges have to get different colors in the second coloring, from which $\binom{n}{2}/k \leq k$ follows, which gives the lower bound.

To prove the upper bound it will be shown that $\chi^2(4, 2^m - 1) \leq 2^m - 1$. Label the vertices of $K_{2^m - 1}$ by non-zero elements $a_1, \dots, a_{2^m - 1}$ of $GF(2^m)$. In the first round, color the edge (a_i, a_j) by (color) $a_i + a_j$. Observe, that for any pair of vertices, $a_i + a_j \neq 0$, since $a_i \neq a_j$. Hence in the first round $2^m - 1$ colors are used. In the second coloring, color the edge (a_i, a_j) with (color) $1/a_i + 1/a_j$. The second round also uses $2^m - 1$ colors. We claim that this two-round coloring is rainbow.

Assume, to the contrary that it is not rainbow. Then there exist four vertices labeled by, say a_1, a_2, a_3, a_4 which span a K_4 that is not rainbow in either of the two colorings. Since for every element in $GF(2^m)$ $a_i = -a_i$, the above assumption means that

$$a_1 + a_2 = a_3 + a_4 \quad \text{and} \quad 1/a_1 + 1/a_2 = 1/a_3 + 1/a_4 \quad (1)$$

simultaneously hold. Let $b_i = a_i/a_4$, $i = 1 \dots 4$. Then the Eqs. (1) imply that

$$b_1 + b_2 + b_3 = 1 \quad \text{and} \quad 1/b_1 + 1/b_2 + 1/b_3 = 1 \quad (2)$$

hold simultaneously. From the first equation of (2) $b_3 = b_1 + b_2 + 1$ follows. Substituting this into the second equation of (2) and multiplying it by $b_1 b_2 b_3$ it follows that

$$\begin{aligned} b_2(b_1 + b_2 + 1) + b_1(b_1 + b_2 + 1) + b_1 b_2 &= b_1 b_2 + b_1^2 b_2 + b_1 b_2^2 \\ b_2 + b_1 b_2 + b_2^2 + b_1 + b_1^2 + b_1 b_2 + b_1 b_2 &= b_1 b_2 + b_1^2 b_2 + b_1 b_2^2 \\ (b_1 + b_2)(b_1 + b_2 + 1) &= b_1 b_2 (b_1 + b_2) \\ 1 + b_1 + b_2 + b_1 b_2 &= (1 + b_1)(1 + b_2) = 0, \end{aligned}$$

which is a contradiction, since $b_4 = 1$ implies neither b_1 nor b_2 equals to 1. ■

Observe that for arbitrary n monotonicity and Theorem 4.1 yield the following.

COROLLARY 4.2. *For $n \geq 4$, $((1 - 1/n)^{1/2}/\sqrt{2}) n \leq \chi^2(4, n) \leq 2n$.*

It would be interesting to determine the correct constant c for which $\chi^2(4, n) = cn + o(n)$. We pose this as an open problem. By the weakness of the considerations in the proof of the lower bound, we are inclined to believe, that $c = 1$. In the following it will be shown that if some special type orthogonal one-factorizations do exist, then $\chi^2(4, n) \leq n - 1$ for an arbitrary even integer n .

Two one-factorizations are orthogonal, if any pair of one-factors—one from the first and one from the second factorization—have at most one edge in common. The existence of pairs of orthogonal one-factorizations for $n \geq 6$ is known, for more details on this topic see the excellent survey paper of Mendelsohn and Rosa [9]. However, to get a rainbow coloring, a property in addition to orthogonality is needed. A pair of orthogonal one-factorizations is C_4 -free, if no $C_4 \subseteq K_n$ has two opposite edges in the same one-factor in the first one-factorization and the other two opposite edges in the same one-factor in the second one-factorization. We are inclined to believe that pairs of orthogonal C_4 -free one-factorizations for every large even n do exist; not being able to prove it, this is posed as an open problem.

PROPOSITION 4.3. *If K_n has a pair of orthogonal C_4 -free one-factorizations, then $\chi^2(4, n) \leq n - 1$.*

Proof. Color an edge of K_n with color i in the first coloring if it is in the i th one-factor of the first factorization, and do the same in the second coloring with respect to the second factorization. Clearly, $1 \leq i \leq n-1$. Assume, that $K_4 \subseteq K_n$ is not rainbow in the first coloring. By the definition of the coloring only pairs of one-factor edges of the given K_4 may get the same color. By orthogonality, in the second coloring those pairs which got the same color in the first coloring will get different colors in the second one. The only problem may occur if a pair of one-factor edges get the same color in the first coloring, and another pair of one factor-edges—which got different colors in the first coloring—will get the same color in the second coloring. But this contradicts our assumption that the pair of one-factorizations used in C_4 -free. ■

The only case where the magnitude of the minimum number of colors needed to rainbow color all possible $K_k \subseteq K_n$ in two rounds is not determined is $k=5$. Clearly, $\chi^2(4, n) \leq \chi^2(5, n)$ so the lower bound obtained in Theorem 4.1 also holds in this case. On the other hand, it was also shown that for infinitely many n $\chi^2(4, n) \leq n$. Despite the fact that the bounds given in the following theorem are not tight, $\chi^2(5, n)$ is separated from $\chi^2(4, n)$ proving a linear lower bound with a coefficient greater than one, while it is separated from $\chi^2(6, n)$ in magnitude, too.

THEOREM 4.4. For $n \geq 5$, $4n/3 - 1 \leq \chi^2(5, n) \leq n^{3/2} + o(n^{3/2})$.

Proof. To prove the lower bound assume that in both colorings the same number of colors is used. Observe, that an arbitrary two-round coloring can be modified into this form, just, say, if in the first round more colors were used, change the colors of some edges of the second coloring one-by-one to new unused colors. Clearly, if the original coloring is rainbow, then the modified one is rainbow also, and the number of colors used in the first coloring remains the same.

If in the first coloring two incident edges (x, y) , (y, z) are colored the same, then in the second one every edge incident to one of vertices $\{x, y, z\}$ has to be colored differently. So in this case in the second round at least $3n-6$ colors have to be used. Therefore, we may assume that color classes are defined by pairwise disjoint edges.

Consider the multigraph G^i containing the (maybe multiple) edges colored by (color) i in one or both colorings. It is easy to see, that a component of G^i may contain at most three edges. Indeed, if some component contains at least four edges, then a subcomponent of it contains four edges. This subcomponent contains two edges colored i in the first, and two edges colored i in the second coloring, since in each coloring the color classes are subsets of one-factors. But these four edges span at most five points and hence the spanned K_5 is not rainbow, a contradiction.

Therefore, each component of G^i contains at most three edges and at least four vertices, or similarly at most two (one) edges and at least three (two) vertices. Therefore, for the number χ of colors used, we get

$$(3n/4) \chi \geq \sum_{i=1}^{\chi} E(G^i) = n(n-1),$$

which gives the lower bound.

In order to prove the upper bound it will be shown that $\chi^2(5, q^2) \leq q^2(q+1)$, where q is an arbitrary power of prime. Again, the density of prime powers will give the upper bound of Theorem 4.4.

Consider the affine plane $AG(2, q)$ and in the first coloring color each of the $\binom{q}{2}$ edges (i.e. pairs of points) of each line of the parallel class i ($i = 1, \dots, q+1$) differently with the same $\binom{q}{2}$ colors. Do the same coloring for each parallel class, but use $\binom{q}{2}$ different colors for each one. Hence, in the first coloring the number of colors used is $(q+1)\binom{q}{2}$. In the second coloring take a one-factorization of a line (i.e. considering it as K_q) and color every edge according to one-factors. Do the same with every line, choosing always different sets of colors for different lines. Hence, in the second coloring the number of colors is $(q-1)q(q+1)$ ($q^2(q+1)$), if q is even (odd), respectively. This two-round coloring is rainbow by the following.

Assume that some K_5 is not rainbow in the first coloring. This means, that edges colored the same are on parallel lines, i.e. at most three of total five points of this K_5 can be on the same line. By the one-factor coloring in second round, edges determined by those vertices will get different colors, and since all other lines get new color sets the given K_5 will be rainbow in the second coloring. ■

Using an observation of Axenovich [1], the lower bound in Theorem 4.4 can be improved to $(1 + \sqrt{5})n/2$, but probably $\chi^2(5, n)$ is superlinear. Thus we pose the following open problem.

Problem 4.5. Is it true, that $\chi^2(5, n)/n \rightarrow \infty$?

The following theorem shows, that for $k \geq 6$ the minimum number of colors needed to rainbow color all $K_k \subseteq K_n$ in two rounds is quadratic.

THEOREM 4.6. For $n \geq 6$, $\binom{n}{2}/2 \leq \chi^2(6, n) \leq n^2/4 + o(n^2)$, i.e. $\chi^2(6, n) \sim n^2/4$.

Proof. Let χ_1 and χ_2 be the number of unique colors—i.e. colors used only once—in the first and second colorings, respectively. We claim that for rainbow colorings $\chi_1 + \chi_2 \geq \binom{n}{2}$ holds which gives the lower bound. Indeed,

assume to the contrary $\chi_1 + \chi_2 < \binom{n}{2}$. Then—by the pigeon hole principle—there is an edge which is not uniquely colored in neither the first, nor in the second coloring. Take this edge and the two edges colored the same in the first and in the second coloring. These three edges are contained in some K_6 , which is not rainbow either in the first or the second coloring, a contradiction.

In order to prove the upper bound it will be shown that $\chi^2(6, q^2 + q + 1) \leq q^4/4 + O(q^3)$, where q is an arbitrary power of prime. Again, by density of prime powers this will give the upper bound of Theorem 4.6.

Let $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_{(q^2+q)/2+1}\}$ and $\mathcal{L}' = \{\ell'_1, \ell'_2, \dots, \ell'_{(q^2+q)/2+1}\}$ be a partition of the lines of projective plane $PG(2, q)$ into two (almost) equal parts. First color every edge (pair of points) being on lines of \mathcal{L} totally differently, while take one factorizations of lines (as K_{q+1-s}) being in \mathcal{L}' and color—as in the proof of Theorem 4.4—every edge according to one-factors, choosing new color classes for every line. Color \mathcal{L} in the second coloring as \mathcal{L}' in the first coloring is colored and vice versa. The maximum number of colors used in first and second round is at most

$$((q^2 + q)/2 + 1) \binom{q+1}{2} + (q+1)(q^2 + q)/2,$$

which is $q^4/4 + O(q^3)$. This two-round coloring is rainbow by the following.

Assume that some K_6 is not rainbow in the first coloring. Edges colored the same must be from the same one-factor of a line from \mathcal{L}' . This means, that four points of this K_6 are on the same line \mathcal{L}' . The K_4 spanned by these four points in the second coloring will be rainbow, and the resulting K_6 obtained by adding two points will also be rainbow, since no other four points can be on the same line. ■

For $k \geq 7$ the minimum of colors to color all $K_k \subseteq K_n$ rainbow in two rounds is exactly known.

THEOREM 4.7. *Let $\alpha = 1$ if $\lfloor n/2 \rfloor$ is odd and $\alpha = 0$ otherwise. Then*

$$\chi^2(7, n) = \binom{\lceil n/2 \rceil}{2} + \lceil n/2 \rceil \lfloor n/2 \rfloor + \lfloor n/2 \rfloor + \alpha.$$

Proof. First note that if in one of the colorings two adjacent edges get the same color, then in the other coloring each edge has to be colored differently. Indeed, if two adjacent edges get the same color, and in the other coloring there are two edges whose color is the same, then a K_7 containing those four edges is not rainbow. Thus we may assume, that only subsets of one factors may get the same color.

Let V_1 (V_2) denote the set of vertices such that all edges adjacent to V_1 (V_2) get unique colors in the first (second) coloring. Then, $V_1 \cup V_2 = V(K_n)$, since if this is not true there is a vertex adjacent to a non-uniquely colored edge in both colorings. A K_7 containing those two pairs of non-uniquely colored edges is clearly not rainbow. Therefore $|V_1| + |V_2| \geq n$. Without loss of generality, we may assume, that $|V_1| \geq \lceil n/2 \rceil$. Since in the first coloring all edges adjacent to V_1 get different colors and the rest of edges can get the same colors only if they are in the same one-factor, we get that the minimum number of colors used in the first coloring is at least

$$\binom{|V_1|}{2} + |V_1| |V_2| + |V_2|,$$

if $|V_2|$ is odd, and

$$\binom{|V_1|}{2} + |V_1| |V_2| + |V_2| - 1,$$

if $|V_2|$ is even. Obviously, those expressions take their minimum choosing $|V_2| = n - |V_1|$ and $|V_1| = \lceil n/2 \rceil$, from which the lower bound follows.

In order to prove the upper bound, partition the set of vertices into two almost equal parts $|V_1| = \lceil n/2 \rceil$ and $|V_2| = \lfloor n/2 \rfloor$. In the first coloring color all edges adjacent to vertices of V_1 differently, and color the rest of edges according to one factors on point set V_2 . In the second coloring do the same just changing V_1 into V_2 and vice versa. Clearly, in the first coloring the number of colors used is at least as many as in the second one and it is as many as in this theorem stated. It remains to show that this two round coloring is rainbow.

Indeed, if some K_7 is not rainbow in the first coloring, then—by one-factor coloring—it contains at least four vertices from V_2 . Similarly, if it is not rainbow in the second coloring, then it contains at least four vertices from V_1 , which contradicts to the fact that V_1 and V_2 are disjoint. ■

Notice, that the proof of Theorem 4.7 can be generalized for more (than two) round colorings, which gives the following.

Remark 4.8. Let $\alpha = 1$ if $\lfloor n/r \rfloor$ is odd and $\alpha = 0$ otherwise. Then

$$\chi^r(4r - 1, n) = \binom{n - \lfloor n/r \rfloor}{2} + n - \lfloor n/r \rfloor \lfloor n/r \rfloor + \lfloor n/r \rfloor + \alpha.$$

Finally, it is easy to show that if $k \geq 8$ then all $\binom{n}{2}$ colors have to be used.

PROPOSITION 4.9. *If $k \geq 8$ then $\chi^2(k, n) = \binom{n}{2}$.*

Proof. It will be shown, that already in case $k = 8$ all $\binom{n}{2}$ colors have to be used. From this the statement for $k \geq 8$ follows by monotonicity.

Assume to the contrary that less colors are used. Then in both colorings there is a pair of edges colored the same. Any K_8 containing those four edges is not rainbow, which is a contradiction. ■

Also note that the proof of Proposition 4.9 can be generalized for more (than two) round colorings, too, which gives the following.

Remark 4.10. If $k \geq 4r$ then $\chi^r(k, n) = \binom{n}{2}$.

5. ANTI-RAMSEY COLORINGS OF LARGE SUBGRAPHS

In previous sections rainbow colorings of small subgraphs of K_n were considered. Here the minimum number of colorings $t(k, n)$ will be studied such that given $\binom{k}{2}$ colors in each coloring, for every $K_k \subseteq K_n$, at least in one of the total t colorings of K_n all $\binom{k}{2}$ edges get different colors. We shall investigate this problem when k increases with n . First we consider the largest possible non-trivial k , i.e. $k = n - 1$. For $n = 4$ the factorization of K_4 makes each triangle rainbow in one round. The next result shows that the situation changes for $n \geq 5$.

THEOREM 5.1. For $n \geq 5$, $t(n - 1, n) = \lceil n/2 \rceil$.

Proof. To prove the upper bound, partition the vertex set of K_n into $\lceil n/2 \rceil$ parts, each containing two vertices, with the last part containing one vertex if n is odd. In the i th coloring color all $\binom{n-2}{2}$ edges spanned by the $n - 2$ points *not* in the i th part differently, and color the $n - 2$ pairs of edges incident to the endpoints of edge in the i th part with $n - 2$ different new colors. Color the edge between the two points in the i th part arbitrarily. The number of colors used is clearly $\binom{n-2}{2} + n - 2 = \binom{n-1}{2}$, and a given K_{n-1} is rainbow colored in the i th coloring if it does not contain the edge in the i th part.

To prove the lower bound, it will be shown, that in each round at most two copies of $K_{n-1} \subseteq K_n$ can be rainbow colored simultaneously. Since there are n different copies in total, this will indicate the desired result.

Let $V(K_n) = \{x_1, x_2, \dots, x_n\}$ and $V(K_{n-1}^i) = \{x_1, x_2, \dots, x_n\} \setminus \{x_i\}$ for $i = 1, \dots, n$. Assume that K_{n-1}^1 and K_{n-1}^2 are rainbow colored in some coloring. Then all edges between x_2 and the vertex set $\{x_3, \dots, x_n\}$ and x_1 and the vertex set $\{x_3, \dots, x_n\}$ have to be colored with (the same) $n - 2$ different colors, say $\{1, \dots, n - 2\}$. For arbitrary $i \geq 3$, K_{n-1}^i will contain all edges between x_1 and $\{x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. The edges between x_1 and $\{x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ are colored with colors $\{1, \dots, n - 2\} \setminus \{j\}$ and

the edges between x_2 and $\{x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ are colored with colors $\{1, \dots, n-2\} \setminus \{k\}$, where j and k are the colors of the edges $\{x_1, x_i\}$ and $\{x_2, x_i\}$, respectively. For $i \geq 3$ K_{n-1}^i contains both x_1 and x_2 . But if $n \geq 5$ then $2(n-3) > n-2$, i.e. for $n \geq 5$ the two color sets $\{1, \dots, n-2\} \setminus \{j\}$ and $\{1, \dots, n-2\} \setminus \{k\}$ will have a non-empty intersection, i.e. for $i \geq 3$ K_{n-1}^i is not rainbow. ■

THEOREM 5.2. *For n sufficiently large, $t(n-2, n) \sim n^2/8$.*

Proof. If $n = 8k + 1$, the edge set of K_n can be decomposed into $\binom{n}{2}/4$ four-cycles. (For more details on this topic see, e.g., the book of Bosák [2].) In the j th coloring color the edges of the j th four-cycle the same, the remaining two edges of the four-cycle arbitrarily and—using $2n - 8$ additional colors—color the edges between the four cycle and the remaining $|V'| = n - 4$ points as follows. If x_1, x_2, x_3, x_4 are the four consecutive points of the four-cycle, then color the edges connecting x_1 and x_3 with V' with the same $n - 4$ additional colors, and do the same with x_2 and x_4 , but with additional $n - 4$ colors. Finally, rainbow color the edges between vertices of V' with an additional $\binom{n-4}{2}$ colors. Hence, the number of colors used is

$$1 + 2n - 8 + \binom{n-4}{2} = \binom{n-2}{2}.$$

Obviously, a coloring induced by a given four-cycle rainbow colors all four $K_{n-2} \subseteq K_n$ containing two consecutive points of the four-cycle and V' . The sequence of above colorings corresponds to all of the $\binom{n}{2}/4$ four-cycles forming a decomposition of K_n is rainbow by the following.

Take an arbitrary $K_{n-2} \subseteq K_n$ and let $\{x_{j_1}, x_{j_2}\} = V(K_n) \setminus V(K_{n-2})$. Then there is a four-cycle of the decomposition—say, $x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}$ which contains this edge. The coloring induced by this four-cycle rainbow colors the given K_{n-2} . Indeed, since x_{j_3} and x_{j_4} are consecutive points of the four-cycle, the (given) K_{n-2} induced by $\{x_{j_3}\} \cup \{x_{j_4}\} \cup V' = V(K_n) \setminus \{x_{j_1}\} \setminus \{x_{j_2}\}$ in this coloring is rainbow. For $n \neq 8k + 1$, the monotonicity implies $t(n-2, n) \leq \binom{n}{2}/4 + O(n)$.

To prove the lower bound, it will be shown, that in each round at most four copies of $K_{n-2} \subseteq K_n$ can be rainbow colored simultaneously. Since there are $\binom{n}{2}$ different copies in total, this will indicate the desired result.

Again, as in the proof of Theorem 5.1, we may assume that each coloring contains at least one rainbow $K_{n-2} \subseteq K_n$. Therefore, in each coloring the number of uniquely colored edges is at least

$$\binom{n-2}{2} - \left(\binom{n}{2} - \binom{n-2}{2} \right) = \binom{n-4}{2} - 4 > \binom{n-5}{2},$$

for $n \geq 10$. This follows from the fact, that there is a $K_{n-2} \subseteq K_n$ whose edges get totally different colors and the colors of the remaining $\binom{n}{2} - \binom{n-2}{2}$ edges may coincide with ones of those totally different ones. Hence, in each coloring i the uniquely colored edges span a graph with vertex set $|V_i| \geq n - 4$ if $n \geq 10$.

Observe that the vertex set of each K_{n-2} rainbow colored in the i th round contains V_i . Indeed, otherwise there is an edge adjacent to some $x \in V_i \setminus K_{n-2}$ which is colored uniquely, i.e. we do not have all the $\binom{n-2}{2}$ colors needed to color the edges of K_{n-2} rainbow.

The remainder of the proof is a case analysis. By the above observations, each V_i is contained in every rainbow K_{n-2} of the i th round, so if $|V_i| = n - 2$, then exactly one $K_{n-2} = V_i$ is rainbow colored. Similarly, if $|V_i| = n - 3$, then at most three copies K_{n-2} are rainbow colored, since V_i can be extended in this case with an additional vertex in only three different ways. Assume, that $|V_i| = n - 4$, and there are at least five copies of K_{n-2} rainbow colored. Then there are three vertices—say x_1, x_2, x_3 —in $V(K_n) \setminus V_i$ such that V_i extended with arbitrary two of those is rainbow. But in order to get all three copies of K_{n-2} spanned by (all possible combinations of) vertices $V_i \cup \{x_j\} \cup \{x_k\}$ ($1 \leq j < k \leq 3$) rainbow, all $3(n - 4)$ edges between V_i and x_j ($j = 1, 2, 3$) and all edges of V_i have to be colored with different colors. Therefore, the number of used colors is

$$\binom{n-4}{2} + 3(n-4) > \binom{n-2}{2}$$

for $n \geq 6$. This contradicts to the assumption that only $\binom{n-2}{2}$ colors were used. ■

Remark 5.3. Observe that for integers of form $n = 8k + 1$ the above result is tight, i.e. for $k \geq 1$

$$t(8k - 1, 8k + 1) = \left(\frac{8k + 1}{2} \right) / 4.$$

Finally, it will be shown, that $t(k, n)$ can also be exponentially large. Since exponential bounds will be proved, the use of integers parts will be omitted to simplify the proof. Note that with the same ideas, everything can be proved without divisibility assumptions.

THEOREM 5.4. *For n sufficiently large $1.008^n \leq t(n/2, n) \leq 1.649^n$.*

Proof. In order to prove the lower bound it will be shown that in each coloring at most

$$8^{n/256} \binom{n - n/128}{n/2 - n/128} \quad (3)$$

copies of $K_{n/2}$ can be rainbow colored. Since there are $\binom{n}{n/2}$ copies in total, the fraction of $\binom{n}{n/2}$ and (3) is a lower bound for the minimum number of rounds of colorings. Standard calculations—e.g. using the Stirling formula—leads to the desired lower bound.

To prove inequality (3), a color (of a fixed coloring) is called sparse if at most eight edges are colored with it. Observe that—by averaging argument—there are at least $\binom{n/2}{2}/2$ sparse colors in the coloring. (Indeed, if $e(i)$ is the number of edges colored i , then $\sum_{i=1}^{\binom{n/2}{2}} e(i) = \binom{n}{2}$, and take the smallest $\binom{n/2}{2}$ terms of this sum.) The edges in sparse color classes cover (at most sixteen) vertices called the span of the color class. Select the maximum number of pairwise vertex disjoint spans S_1, S_2, \dots, S_t from the sparse color classes, let S denote their union. Without loss of generality assume that S_i is spanned by the sparse color i for $1 \leq i \leq t$. We claim that $|S| \geq n/16$.

Indeed, assume that $|S| = \varepsilon n$. From the definition of S , this means that the

$$\binom{\varepsilon n}{2} + \varepsilon n(1 - \varepsilon)n \quad (4)$$

edges incident to S contain at least one edge of each sparse color. But for $\varepsilon \leq 1/16$ and n sufficiently large the sum (4) is less the $\binom{n/2}{2}/2$ contradicting to our starting observation. Hence the claim holds.

Since $|S_i| \leq 16$, the claim implies that $t \geq |S|/16 \geq n/256$. Setting $m = \lfloor n/256 \rfloor$, the number of rainbow copies of $K_{n/2}$ in a given coloring can be estimated as follows. A rainbow $K_{n/2} \subset K_n$ must contain every color, since the number of colors and edges is equally $\binom{n/2}{2}$. Therefore, it must also contain edges of color i for $1 \leq i \leq m$ and these can be selected in at most 8^m ways from the spans S_i . Since these spans are vertex disjoint, we have selected $2m$ vertices thus the remaining $n/2 - 2m$ vertices must be selected from the remaining $n - 2m$ vertices of K_n . This argument shows that there are at most $8^m \binom{n - 2m}{n/2 - 2m}$ rainbow copies of $K_{n/2} \subset K_n$ which (apart from the negligible problem of using $m = \lfloor n/256 \rfloor$ instead of $n/256$) gives inequality (3) and the lower bound.

The trivial consideration gives only a $t(n/2, n) \leq \binom{n}{n/2} \approx 2^n$ upper bound. To make an exponential improvement just observe that if a set of partitions of vertices of K_n into $k \leq n$ parts is given such that vertices of each $K_k \subseteq K_n$

are in different parts in at least one of the partitions, then the number of partitions $|\mathcal{P}|$ is an upper bound for $t(k, n)$.

Indeed, in the i th ($i = 1, \dots, |\mathcal{P}|$) coloring, color the edges between two different parts the same, but for all $\binom{k}{2}$ possibilities with different colors. Edges within the same part color arbitrarily. A given $K_k \subseteq K_n$ will be rainbow in the coloring associated to the partition in which all k vertices of K_k are in different parts.

Let $P(k, n)$ denote the minimum size of such a set of partitions. To obtain the upper bound one can directly use the Fredman–Komlós (upper) bound for the minimum size of a (k, k) family of perfect hash functions [5]. To be self contained we present here an upper bound for $P(k, n)$ based on a lemma of Erdős and Kleitman [3] which says that every k -uniform family \mathcal{F} contains a k -partite subfamily $\mathcal{F} \subseteq \mathcal{F}'$ of size $|\mathcal{F}'| \geq (k!/k^k) |\mathcal{F}|$.

Let \mathcal{F}_0 be the hypergraph containing all $\binom{n}{k}$ k -tuples of vertices of K_n . Let \mathcal{F}'_0 be as in the above lemma, i.e. a k -partite subfamily of \mathcal{F}_0 containing (by Erdős–Kleitman) at least $(k!/k^k) |\mathcal{F}_0|$ k -sets. Let $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}'_0$ and use the Erdős–Kleitman lemma for \mathcal{F}_1 . In general, if \mathcal{F}_i is given, use the Erdős–Kleitman lemma for it and getting \mathcal{F}'_i define $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{F}'_i$. By the Erdős–Kleitman lemma after the i th step the number of k -tuples which are not k -partite in any of the first i \mathcal{F}'_i -s is at most

$$(1 - k!/k^k)^i \binom{n}{k}. \quad (5)$$

Clearly, $P(k, n) \leq i$ for the smallest i which makes formula (5) less than one. Using Stirling formula, it follows, that

$$(1 - k!/k^k)^i \binom{n}{k} \leq (1 - 1/e^k)^i \binom{n}{k} \leq (1/e^{i/e^k}) \binom{n}{k} \leq 1,$$

if $i \geq e^k \ln \binom{n}{k}$. Therefore,

$$t(k, n) \leq P(k, n) \leq e^k \ln \binom{n}{k}, \quad (6)$$

which in the case $k = n/2$ for $t(n/2, n)$ gives a (roughly) $(\sqrt{e})^n$ upper bound. The upper bound of Theorem 5.4 follows from this. ■

Observe that the proof technique for the upper bound in Theorem 5.4 is useful for arbitrary $k \leq n$. In particular, combining this with the proof method of Körner and Simonyi for the lower bound of $t(3, n)$ we get the following.

COROLLARY 5.5. *If k is constant, then $t(k, n) = \Theta(\log n)$.*

Proof. The upper bound follows from (6) and to get the lower bound observe (as in Theorem 3.2) that $(n-1)/\binom{k}{2}^t \leq 1$.

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REFERENCES

1. M. Axenovich, personal communication.
2. J. Bosák, "Decompositions of Graphs," Kluwer Academic, Bratislava, 1990.
3. P. Erdős and D. J. Kleitman, On coloring graphs to maximize the proportion of multi-colored k -edges, *J. Combin. Theory* **5** (1968), 164–169.
4. P. Erdős, M. Simonovits, and V. T. Sós, Anti-Ramsey theorems, in "Infinite and Finite Sets, Keszthely, Hungary," Colloq. Math. Soc. János Bolyai, Vol. 10, pp. 633–642, North-Holland, Amsterdam, 1973.
5. M. Fredman and J. Komlós, On the size of separating systems and perfect hash functions, *SIAM J. Algebraic Discrete Methods* **5** (1984), 61–68.
6. M. Hall, "Combinatorial Theory," Wiley-Interscience, New York, 1986.
7. J. Körner and K. Marton, New bounds for perfect hashing via information theory, *European J. Combin.* **9** (1988), 523–530.
8. J. Körner and G. Somonyi, Trifférance, *Studia Sci. Math. Hungar.* **30** (1995), 95–103.
9. E. Mendelsohn and A. Rosa, One-factorizations of the complete graph—a survey, *J. Graph. Theory* **9** (1985), 43–65.
10. M. Simonovits and V. T. Sós, On restricted colourings of K_n , *Combinatorica* **4** (1984), 101–110.