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ON-LINE 3-CHROMATIC GRAPHS I. TRIANGLE-FREE GRAPHS*

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Abstract. This is the first half of a two-part paper devoted to on-line 3-colorable graphs. Here on-line 3-colorable triangle-free graphs are characterized by a finite list of forbidden induced subgraphs. The key role in our approach is played by the family of graphs which are both triangle- and $(2K_2 + K_1)$-free. Characterization of this family is given by introducing a bipartite modular decomposition concept. This decomposition, combined with the greedy algorithm, culminates in an on-line 3-coloring algorithm for this family. On the other hand, based on the characterization of this family, all 22 forbidden subgraphs of on-line 3-colorable triangle-free graphs are determined. As a corollary, we obtain the 10 forbidden subgraphs of on-line 3-colorable bipartite graphs. The forbidden subgraphs in the finite basis characterization are on-line 4-critical, i.e., they are on-line 4-chromatic but their proper induced subgraphs are on-line 3-colorable. The results of this paper are applied in the companion paper [Discrete Math., 177 (1997), pp. 99–122] to obtain the finite basis characterization of *connected* on-line 3-colorable graphs (with 51 4-critical subgraphs). However, perhaps surprisingly, connectivity (or the triangle-free property) is essential in a finite basis characterization: there are infinitely many on-line 4-critical graphs.

Key words. on-line coloring, forbidden subgraphs

AMS subject classifications. 05C15, 05C75, 05C85

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Introduction. A *proper coloring* of a graph G is an assignment of positive integers (called colors) to its vertices in such a way that adjacent vertices have distinct colors. The smallest number of colors in any proper coloring is denoted by $\chi(G)$ and is called the *chromatic number* of G. An *on-line coloring* of a (finite) G is an algorithm that colors the vertices as follows:

- Vertices of G are given in some order $v_1, v_2, \ldots$ (unknown by the algorithm).
- In the ith step the algorithm assigns a proper color to v_i (and never changes it later).

The most extensively studied on-line coloring algorithm is the greedy or *first fit* algorithm (FF): in each step it assigns the smallest available positive integer as color to the current vertex. In general, on-line coloring can be interpreted as a two-person game of GraphDrawer and GraphPainter. Drawer's moves consist of successively revealing vertices of a graph G with all adjacencies to vertices already known by Painter, and in each step Painter assigns a color to the current vertex. Painter's aim is to use as few distinct colors as possible while Drawer's aim is to force Painter to use as many colors as possible. The common optimum value will be called the on-line chromatic number of G.

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Let G be a graph and A be some fixed on-line coloring algorithm. Then the maximum number of colors used by A during any coloring game (i.e., for all orderings of the vertices of G) is called the A-chromatic number of G and is denoted by $\chi_A(G)$. The *on-line chromatic number*, $\chi^*(G)$, is the minimum number of colors Painter succeeds with when playing on G; that is, $\chi^*(G) = \min\{\chi_A(G) : A \text{ is an on-line coloring}\}$. A graph G is (on-line) *k-critical* if $\chi^*(G) = k$ and $\chi^*(G') < k$ holds for every proper induced subgraph $G' \subset G$.

The concept of on-line chromatic number of graphs was introduced in [GL1], [GL2]; a similar notion, recursive coloring, had been investigated earlier. The introduction in [KPT1] gives a brief survey of the connection of these concepts. Our reference list covers several areas of on-line graph colorings beyond our particular subject [GKL2], [I], [K], [K1], [K2], [KK], [KT1], [KT], [LST], [V].

On-line 2-colorable graphs are rather trivial, and their connected components are complete bipartite graphs. This statement is a good introductory exercise to on-line colorings. It also shows that a single on-line algorithm, FF, provides a 2-coloring for every on-line 2-colorable graph. This is not the case for on-line 3-colorable graphs as demonstrated by the **B**-**E** paradigm [GL2]: although the graphs **B** and **E** (see Figure 1) are on-line 3-colorable, Painter cannot color with three colors if Drawer does not tell in advance which graph is to be presented. Thus a single on-line 3-coloring algorithm cannot 3-color every on-line 3-colorable graph. The same phenomenon explains that such a simple operation as addition of an isolated vertex may change on-line 3-colorability of a graph. The smallest amusing example is the triangle with a pendant edge on each of its vertices [GKL1]. A bipartite example comes from the evolution of **B**. Adding an isolated edge and an isolated vertex to the graph **B** gives an on-line 3-colorable graph, but if a further isolated vertex is added, an on-line 4-chromatic graph is obtained. These examples might suggest that on-line 3-colorable graphs are very restricted, but examples like the Petersen graph, $K_3 \times K_3$ [GKL1], seem to refute this view. It seems to us that the analysis of on-line 3-colorable graphs is a good test case by which to understand paradoxical features of on-line colorings. As pointed out by referees, our approach is tailored specifically to 3-colorable graphs and at many places relies heavily on case analysis. Unfortunately, this seems to be an inherent feature of the subject.

This paper gives a characterization of on-line 3-colorable triangle-free graphs. The crucial role is played by the family of graphs which are both triangle- and $(2K_2+K_1)$-free. We use the notation $(\mathbf{\Delta}, \mathbf{\Xi})$-free for this family in accordance with our notations $\mathbf{\Delta}$ for the triangle C_3 and $\mathbf{\Xi}$ for $2K_2 + K_1$. Our key result (Theorem 1) states that $(\mathbf{\Delta}, \mathbf{\Xi})$-free graphs are on-line 3-colorable—in fact, with a single on-line algorithm $\mathcal{A}$ (section 3).

Theorem 1 is related to coloring results on $(\mathbf{\Delta}, T)$-free graphs. A well-known conjecture [G], [S] states that $(\mathbf{\Delta}, T)$-free graphs have bounded chromatic number in terms of the number of vertices of T, where T is a forest. The on-line version behaves differently; in [GL1] it was shown that the on-line chromatic number of $(\mathbf{\Delta}, P_6)$-free graphs is not bounded. Sumner proved that $(\mathbf{\Delta}, P_5)$-free graphs are 3-colorable [S] and in fact are 3-colorable by FF as shown in [GL3]. A well-known example (the bipartite complement of mK_2) demonstrates that the FF-chromatic number is unbounded for our $(\mathbf{\Delta}, \mathbf{\Xi})$-free (even for $(\mathbf{\Delta}, K_2 + 2K_1)$-free) family. Thus the on-line 3-coloring algorithm $\mathcal{A}$ of Theorem 1 cannot be replaced by FF. Actually, $\mathcal{A}$ seems to be the first algorithm essentially different from FF which is optimal for a family where FF behaves very poorly. It is worth noting that, going a step further, the family of

$(\mathbf{\Delta}, 3K_2)$-free graphs are not 3-colorable even off-line since the Grötzsch graph is in the family. Finally we note that (prepared by works in [GL2], [GL3], [KPT]) a deep theorem of Kierstead, Penrice, and Trotter [KPT1] implies that the family of $(\mathbf{\Delta}, T)$-free graphs has a bounded on-line chromatic number if and only if each component of the forest T is P_6-free.

Structural and coloring properties of $(\mathbf{\Delta}, \Xi)$-free graphs are interrelated. On one hand, algorithm $\mathcal{A}$ is used to prove structural results; for example, the existence of $\mathcal{A}$ immediately implies (through the **B**-**E** paradigm) that $(\mathbf{\Delta}, \Xi)$-free graphs cannot contain both **B** and **E**. On the other hand, algorithm $\mathcal{A}$ is based on our structural characterization of the family.

To obtain a general structure theorem (Theorems II and 2) we shall introduce a modular decomposition of Ξ-free bipartite graphs in section 2. The building blocks (modules) are $2K_2$-free bipartite graphs (halfgraphs), and they are joined using complete bipartite graphs. Nonbipartite members of the family are obtained by extending bipartite ones having at most two modules, and their structure shows a peculiar circular symmetry (Theorem 1 and (2.7)). This is a graph theoretic structure theorem independent of on-line coloring and so has its own interest.

In section 4.2 we extend algorithm $\mathcal{A}$ to color disconnected $\mathbf{\Delta}$-free graphs containing **B** with three colors when it is possible.

A synthesis of our techniques results in a characterization of on-line 3-colorable triangle-free graphs by finitely many (22) forbidden subgraphs (Theorem 4). In fact, these are the triangle-free on-line 4-critical graphs displayed in Figures 3, 4, and 5 (except F_1 and F_5). We have learned that the Drawer-Painter game is rather interesting on almost all of them due to diverse strategies with subtle details. During a game on any of these graphs, a smart Painter has a chance to achieve a 3-coloring against an imperfect Drawer. However, a perfect Drawer can always force any Painter to use four colors.

Theorem 4 implies that on-line 3-colorability of a triangle-free graph can be decided (theoretically) in polynomial time of its order, in contrast with off-line 3-colorability which is known to be NP-complete [L].

In the companion paper [GKL1] Theorems 2 and 3 were used to obtain the finite basis characterization of *connected* on-line 3-colorable graphs (with 51 forbidden on-line 4-critical subgraphs). In contrast to our expectations, the assumption of connectivity was essential: we found an infinite family of (disconnected) on-line 4-critical graphs. Therefore, on-line 3-colorable graphs (like off-line 2-colorable, i.e., bipartite graphs) cannot be characterized with finitely many forbidden subgraphs.

We conclude the introduction with remarks concerning algorithmic aspects of our results. The structural properties of on-line 3-colorable graphs developed in this paper and in its companion led to a very simple on-line coloring algorithm (FF(C_6) in [GKL1]). This algorithm is a slight modification of FF, easy to implement, and uses at most four colors on every on-line 3-colorable graph. Due to the **B**-**E** paradigm, this is the best that a single on-line algorithm can achieve. Another algorithm for the same purpose, List First Fit, was found independently by Kolossa [KO]. Vaguely speaking, both algorithms are fast optimal, but it is extremely difficult to prove that they do what they claim. Our attempt to sacrifice accuracy for clarity and the hope of generalization led to an on-line algorithm for which it is easy to bound the maximum number of colors (142) for any on-line 3-colorable input graph. Unfortunately, for $k > 3$, the proof is not suitable to give an affirmative answer for the following more general and seemingly important question. For fixed k, is it possible to find a single

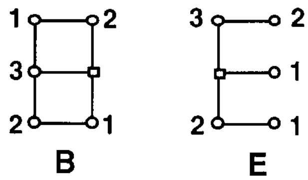

FIG. 1. *Graphs* **B** *and* **E**.

on-line coloring algorithm A_k which colors every on-line k-colorable graph with a bounded number of colors (in terms of k)? [GKL3].

1. Notations and results. Let K_n, P_n, and C_n denote the n-clique, the induced path with n vertices, and the induced n-cycle, respectively. For a positive integer k, kG is the union of k disjoint copies of G and $G+H$ is the disjoint union of the graphs G and H. We use the following nonstandard notation: $\mathbf{II} = 2K_2$, $\mathbf{\Xi} = \mathbf{II} + K_1$, $\mathbf{B}$ is a 6-cycle together with a long chord, and $\mathbf{E}$ is the graph obtained from $\mathbf{B}$ by removing two consecutive edges from its 6-cycle adjacent to the long chord (see Figure 1). The triangle is often denoted by $\mathbf{\Delta}$. Graphs with more than one forbidden subgraph are indicated by the list of subgraphs within parentheses.

The main result of the paper is the following theorem.

THEOREM 1. *If G is $(\mathbf{\Delta}, \mathbf{\Xi})$-free, then $\chi^*(G) \leq 3$. In addition, a 3-coloring for all $(\mathbf{\Delta}, \mathbf{\Xi})$-free graphs is obtained by a single on-line algorithm, $\mathcal{A}$.*

The proof of Theorem 1 is presented in sections 2 and 3. In section 2 we prove a structure theorem for $(\mathbf{\Delta}, \mathbf{\Xi})$-free graphs by introducing a new modular decomposition concept. Section 3 concludes the proof of Theorem 1 by presenting the on-line 3-coloring algorithm $\mathcal{A}$, a combination of FF and a natural but not simple algorithm based on the structure theorem.

Structural characterization of $(\mathbf{\Delta}, \mathbf{\Xi})$-free graphs is developed in several stages. First, $\mathbf{II}$-free members of the family are described (see (2.2) and (2.3)). Next, bipartite $\mathbf{\Xi}$-free graphs are characterized using a modular decomposition technique. The decomposition relies on the fact that a bipartite graph G is $\mathbf{\Xi}$-free if and only if every connected component of the bipartite complement of G contains no $\mathbf{II}$ (see (2.4)). Finally we give extension rules by which all nonbipartite members of the family are derived from bipartite ones (see (2.6) and (2.7)). We summarize here the conclusion of section 2 without explaining the definitions in details. (These can be found at the end of the present section and throughout section 2.)

THEOREM I. *A $\mathbf{\Delta}$-free graph G with no equivalent vertices is $\mathbf{\Xi}$-free if and only if G satisfies one of the following properties:*

(a) $G = C_5 + K_1$.

(b) *G is a bipartite graph such that its bipartite complement is the disjoint union of connected $\mathbf{II}$-free bipartite graphs (called reduced halfgraphs).*

(c) *G is the induced subgraph of a graph H with the following structure: The vertices of H are partitioned into six nonempty sets $A_{i,j}$, $1 \leq i \leq 3, 1 \leq j \leq 2$, such that the graph induced by A_{i_1,j_1} and A_{i_2,j_2} is a complete bipartite graph, if $i_1 = i_2, j_1 \neq j_2$; a halfgraph or a reduced halfgraph, if $i_1 \neq i_2, j_1 = j_2$; and a graph with no edges otherwise. Furthermore, for any $x \in A_{1,j}, y \in A_{2,j}, z \in A_{3,j}$ the set $\{x, y, z\}$ induces neither a triangle nor the complement of a triangle.*

The coloring result of Theorem 1 leads to the following theorem.

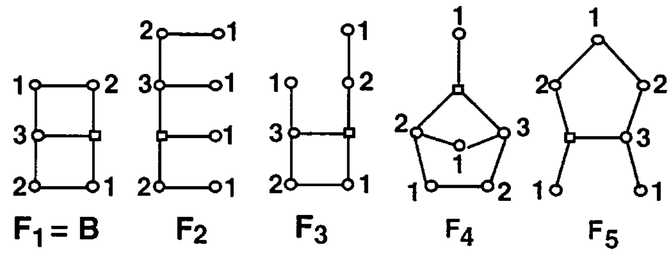

FIG. 2. *The five minimal* $\mathbf{\Delta}$*-free graphs of* FF*-chromatic number* 4.

THEOREM II. *Let* G *be a connected* $\mathbf{\Delta}$*-free graph containing a copy of* $\mathbf{B}$. *Then* G *is* $\mathbf{E}$*-free if and only if* G *is* Ξ*-free.*

For this purely graph theory statement we could not find a short direct proof that avoids on-line colorings. Actually, Theorem II is proved in the following stronger form in section 3.1.

THEOREM 2. *If* G *is a connected* $\mathbf{\Delta}$*-free graph containing a copy of* $\mathbf{B}$, *then the following statements are equivalent:*

(1) G *is* $\mathbf{E}$*-free.*
(2) G *has no induced subgraph isomorphic to any of* F_3, F_4 *in Figure* 2 *and* B_1, B_2, B_3, B_4 *in Figure* 4.
(3) G *is* Ξ*-free.*
(4) G *has on-line chromatic number* $\chi^*(G) \leq 3$.

Theorem 2 also helps in finding the list of all minimal graphs that are excluded from graphs of on-line chromatic number 3. Before formulating this result in Theorem 3 we present some critical graphs from the list. Let us start with the observation that any graph G of on-line chromatic number 4 must contain an induced subgraph G' such that $\chi_{FF}(G') = 4$. In [GKL1] we determined all graphs with FF-chromatic number 4 which are minimal with respect to that property. From the list of these 22 graphs, Figure 2 shows the $\mathbf{\Delta}$-free ones.

In [GKL1] it was also shown that F_2, F_3, and F_4 are 4-critical, $F_1 = \mathbf{B}$ and F_5 are not. Hence, if G is a $\mathbf{\Delta}$-free 4-critical graph different from F_2, F_3, and F_4, then G contains at least one of $\mathbf{B}$ and F_5. Figure 3 shows all 4-critical graphs obtained in [GKL1] which contain F_5.

The analysis of 4-critical graphs results in the following finite basis theorems.

THEOREM 3. *If* G *is a* $\mathbf{\Delta}$*-free graph containing* $\mathbf{B}$, *then* G *has on-line chromatic number at most* 3 *if and only if* G *has no induced subgraph isomorphic to any of* F_3, F_4 *in Figure* 2 *and* B_i, $1 \leq i \leq 10$, *in Figure* 4.

THEOREM 4. *A* $\mathbf{\Delta}$*-free graph* G *has on-line chromatic number* 3 *if and only if* G *has no induced subgraph isomorphic to any of* F_2, F_3, F_4 *in Figure* 2 *and the* 19 *graphs in Figures* 3 *and* 4.

A corollary of Theorem 4 is the following finite basis result for bipartite graphs.

THEOREM 5. *A bipartite graph* G *has on-line chromatic number at most* 3 *if and only if* G *has no induced subgraph isomorphic to* F_2, F_3 *in Figure* 2 *and* $B_1, B_2, B_3, B_5, B_7, B_8, B_9, B_{10}$ *in Figure* 4.

The vertex and the edge set of a graph G is respectively denoted by $V(G)$ and $E(G)$. The relation $D \subset G$ means that D is an induced subgraph of G. Throughout the paper *subgraph* always means *induced subgraph* (i.e., "G has a P_4" actually means that P_4 is an induced subgraph of G). For $D \subset G$ and $v \in V(G)$, $D + v$ and $D - v$

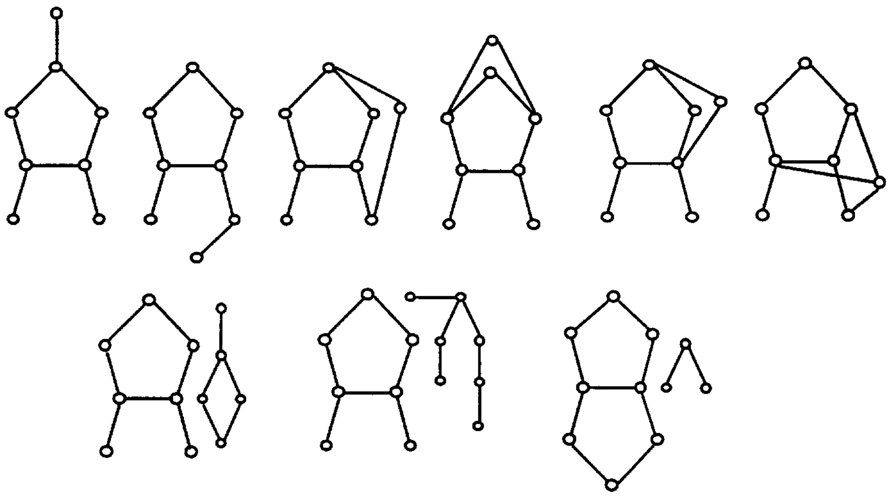

FIG. 3. *All* $\mathbf{\Delta}$*-free* 4*-critical graphs containing* F_5.

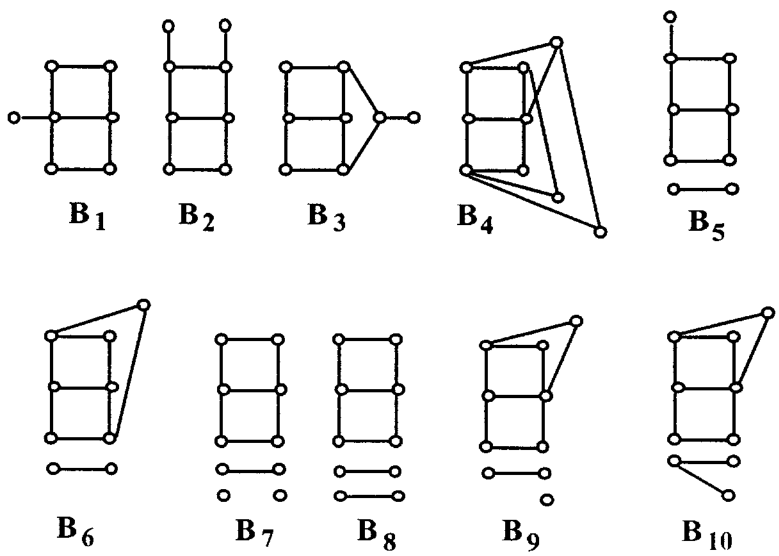

FIG. 4. *All* $\mathbf{\Delta}$*-free* 4*-critical graphs containing* $\mathbf{B}$.

denote the subgraph of G induced by $V(D) \cup \{v\}$ and $V(D) \setminus \{v\}$, respectively.

Define $N(v) = \{u \in V(G) : uv \in E(G)\}$ (and $N_U(v) = \{u \in U \subseteq V(G) : uv \in E(G)\}$) to be the ($U$-) neighborhood set of $v \in V(G)$. Vertices $u, v \in V(G)$ are called *equivalent* if and only if $N(u) = N(v)$. A graph is called *primitive* if it contains no equivalent pair of vertices. Vertex multiplication is the operation of replacing a vertex x of a graph with a certain number of equivalent copies of x. If a graph G is the vertex multiplication of some primitive graph G' then we say that G' is a *primitive representative* of G. For a graph G and $v \in V(G)$, let $\mathcal{C}_G(v) \subset V(G)$ denote the set of all vertices of G equivalent to v. Obviously, any subgraph of G induced by the set containing one vertex from each equivalence class is a primitive representative of G.

Throughout the paper it is assumed that bipartite graphs are given together with a bipartition of their vertices. A bipartite graph with partite sets X and Y is denoted by $[X, Y]$ and is called here a *bigraph*. General graphic operations have natural bipartite versions for bigraphs. In case of bigraphs the equivalence and the vertex multiplication are involving vertices in the same partite set. A bigraph is called primitive if it contains no equivalent pair of vertices. (A primitive bigraph can have two isolated vertices, one in each class.) For a bigraph $G = [X, Y]$, let $\widehat{G}$ denote the bipartite complement of G, that is, $x \in X$ and $y \in Y$ are adjacent in $\widehat{G}$ if and only if xy is not an edge of G. A vertex v is called a *star vertex* of the bigraph $[X, Y]$ if $N(v) = X$ or Y. A subgraph of an arbitrary graph G induced by two disjoint independent sets $X, Y \subset V(G)$ is a bigraph $[X, Y]$ of G.

2. Characterizations of $(\mathbf{\Delta}, \mathbf{\Xi})$-free graphs. Graph classes we shall consider in this section are closed under vertex multiplication. (It is worth noting that this is not true for the whole class of on-line 3-colorable graphs.) This property is formulated in our first proposition. (The trivial proof is omitted.)

(2.1) Let H be a primitive (bi)graph. Then G is an H-free (bi)graph if and only if any primitive representative of G is H-free. $\square$

A $\mathbf{\Xi}$-free graph G with no triangle is of one of the following two types: either G is **II**-free or G contains **II** and is disconnected (Type 1), or G contains **II** and is connected (Type 2).

Type 1. Let G be a graph of Type 1. No connected component of G may contain **II**; otherwise, G would be connected, which contradicts the definition of Type 1 graphs. If G has two nontrivial connected components, then it contains **II**; thus no third component might exist. Because both components must be $(K_2 + K_1)$-free, G is bipartite, and it is the disjoint union of two complete bipartite graphs. Assume next that G has exactly one nontrivial connected component, that is, G is **II**-free.

First let G be a nonbipartite graph of Type 1. Since G is $\mathbf{\Delta}$-free with no **II**, its shortest induced odd cycle must be a C_5. This observation combined with (2.1) results in the following easy characterization.

(2.2) A nonbipartite $\mathbf{\Delta}$-free graph G is of Type 1 if and only if G is the vertex multiplication of C_5 or $C_5 + K_1$. $\square$

Next let G be a bipartite graph of Type 1 containing one nontrivial connected component, or, equivalently, let $G = [X, Y]$ be a **II**-free bigraph. The following four properties are obviously equivalent:

(i) $G = [X, Y]$ is **II**-free;

(ii) for every $x, x' \in X$, either $N(x) \subseteq N(x')$ or $N(x') \subseteq N(x)$;

(iii) X has an ordering $x_1, \ldots, x_p$ such that $Y \supseteq N(x_1) \supseteq \cdots \supseteq N(x_p)$;

(iv) Y has an ordering $y_1, \ldots, y_q$ such that $N(y_1) \subseteq \cdots \subseteq N(y_q) \subseteq X$.

The equivalence of (i) and (iii) characterizes **II**-free bigraphs as follows: $G = [X, Y]$ is **II**-free if and only if $\{N(x) : x \in X\}$ defines a chain on Y (and $\{N(y) : y \in Y\}$ defines a chain on X). The chain on X may start with the empty set (corresponding to an isolated vertex of Y); it may contain several copies of the same subset (corresponding to equivalent vertices of Y), and its last member is either the whole set X (which corresponds to a star vertex in Y) or the set of nonisolated vertices in X.

Using these observations together with (2.1), all **II**-free bigraphs can be obtained from the containment graphs of simple chains, called halfgraphs. The nth halfgraph,

$H(n)$, is defined as a bigraph on vertex set $\{x_1,\dots,x_n\}\cup\{y_1,\dots,y_n\}$ with x_iy_j being an edge if and only if $i<j$. Notice the symmetry of $H(n)$ defined by the automorphism $x_i \longleftrightarrow y_{n+1-i}$ $(i=1,\dots,n)$ between its partite sets. In this paper we call a bigraph halfgraph if it is a vertex multiplication of some $H(n)$. The primitive halfgraphs will be written as halfgraphs. A halfgraph which does not have isolated vertices in both bipartition classes will be called a *reduced halfgraph*.

(2.3) A bigraph G is of Type 1 if and only if G is the vertex multiplication of $\mathbf{II}$ or it is a halfgraph or a reduced halfgraph. □

The type of Ξ-free bigraphs can be determined by introducing a new modular decomposition concept which also will be useful for the whole structural description of $(\mathbf{\Delta},\Xi)$-free graphs. Observe first that the bipartite complement of the bigraph Ξ is a P_5. Hence a bigraph G is Ξ-free if and only if its bipartite complement $\widehat{G}$ is P_5-free. Furthermore, a connected component of $\widehat{G}$ contains no P_5 if and only if it is $\mathbf{II}$-free. According to the discussion before (1.3) each connected $\mathbf{II}$-free bigraph is a connected reduced halfgraph, that is, either some isolated vertices or a halfgraph with all of its isolated vertices removed. Note that $\widehat{\mathbf{II}}\cong\mathbf{II}$; hence the connected components of $\widehat{\mathbf{II}}$ are isomorphic to K_2 (i.e., a connected reduced $H(2)$).

(2.4) A bigraph is Ξ-free if and only if every connected component of its bipartite complement is a connected reduced halfgraph. □

Let G be a Ξ-free bigraph and denote by $G_1,\dots,G_k$ the connected components of its bipartite complement $\widehat{G}$. Then $\widehat{G_i}$, $i=1,\dots,k$, are called the *modules* of G. If a module contains just one vertex, then it is called a *trivial module*; otherwise, it is a *nontrivial module*. Observe that any two vertices from distinct partite sets and from distinct modules are adjacent in G; in particular, trivial modules are star vertices of the bigraph. It is easy to check that the bipartite complement of a connected reduced halfgraph is either a single vertex or a halfgraph. Therefore, by (2.3), *each nontrivial module of the unique module decomposition of G is a halfgraph*. Note that the two modules of the bigraph $\mathbf{II}$ are isomorphic to $H(1)$. Thus we obtain that a Ξ-free bigraph G is of Type 1 if and only if G has $k\le 2$ nontrivial modules and, in case of $k=2$, neither contains an edge.

Type 2. As a result of the module decomposition concept introduced for Ξ-free bigraphs we obtain that a bigraph G is of Type 2 if and only if G is connected and has $k\ge 2$ nontrivial modules. Nonbipartite graphs of Type 2 will be described as extensions of Ξ-free bigraphs.

Let G be a graph, $D\subset G$, and $z\in V(G)\setminus V(D)$. If D is a bigraph and $D+z$ is also bipartite, then z is called a *bipartite extension* of D. If $D+z$ is nonbipartite (i.e., z induces an odd cycle together with some vertices of D), then z is called an *odd extension* of D. The obvious transition rule of bipartite extensions are described as follows.

(2.5) Let G be a $(\mathbf{\Delta},\Xi)$-free graph and let $D=[X,Y]$ be a connected induced bigraph of G. If $z\in V(G)\setminus V(D)$ is a bipartite extension of D and M is the module of $D+z$ containing z, then $M=\{z\}$, or $M-z$ is a module of D, or $M-z$ consists of at most one nontrivial module and a set of trivial modules of D. □

For characterizing nonbipartite graphs of Type 2 we need to extend the notion of halfgraphs. Let $F=[X,Y]$ be a halfgraph and let z be a new vertex adjacent to some vertices of F, i.e., z is an extension of F with neighborhood sets $X(z)\subseteq X$ and

$Y(z) \subseteq Y$. The graph $F + z$ is called an *extended halfgraph* if the following properties are all satisfied:

- $X(z) \neq \emptyset$ and $Y(z) \neq \emptyset$.
- $F + z$ is $\mathbf{\Delta}$-free.
- If $x \in X,\ y \in Y,\ xy \notin E$, then at least one of zx and zy is an edge.

The second and third properties together say that there are neither triangles nor empty triangles of form zxy.

The following statement describes the structure of nonbipartite graphs of Type 2.

(2.6) Let G be a connected $(\mathbf{\Delta}, \mathbf{\Xi})$-free graph and let $D = [X, Y]$ be an induced bigraph of G. Assume that the set $Z \subset V(G) \setminus V(D)$ of all odd extensions of D is nonempty. Then D and Z satisfy (i)–(iv):

(i) D has at most two nontrivial modules. Furthermore, for any fixed $z \in Z$, the neighbors of z in D belong to the same nontrivial module.

(ii) If $M_1 \subset D$ is the module containing the neighbors of $z \in Z$, then $M_1 + z$ is an extended halfgraph.

(iii) If $z_1, z_2 \in Z$ are distinct, then $z_1 z_2 \in E(G)$ if and only if z_1 and z_2 are adjacent to distinct nontrivial modules of D.

(iv) Let $M_1 = [X_1, Y_1] \subset D$ be a module and $Z_1 = \{z \in Z : \text{neighbors of } z \text{ are in } M_1\}$, and let A, B, C denote the sets X_1, Y_1, Z_1 in any order and $c_1, c_2 \in C$. Then either $N_A(c_1) \subseteq N_A(c_2)$ or $N_A(c_2) \subseteq N_A(c_1)$. Moreover, if $N_A(c_1) \subset N_A(c_2)$, then $N_B(c_2) \subseteq N_B(c_1)$.

Proof. Recall that every nontrivial module of D is a halfgraph and its trivial modules are star vertices. Because G is $\mathbf{\Xi}$-free with no triangle, all induced odd cycles of $D + z \subset G$ are isomorphic to either C_5 or C_7. Because z is an odd extension, at least one induced odd cycle containing z must exist in $D + z$.

Because G is connected, every nonbipartite subgraph of G must be connected (otherwise, as easily can be checked, G would contain $\mathbf{\Xi}$). In particular, $D+z$ is connected for every $z \in Z$.

(i) Let $z \in Z$ and assume that C is an induced odd cycle of $D + z$ with $x \in V(C) \cap X(z)$ and $y \in V(C) \cap Y(z)$. Because $xy \notin E(G)$, x and y are vertices of the same module, say, $M_1 = [X_1, Y_1]$. If $u \in V(D)$ is a vertex not in M_1, then one of ux and uy is an edge of G; thus $zu \notin E(G)$ follows (because G is $\mathbf{\Delta}$-free). This shows that the neighbors of z in D belong to M_1. Assuming that D has more than two nontrivial modules, a copy of $\mathbf{II}$ between M_2 and M_3 together with z would induce a $\mathbf{\Xi}$ of G. Thus D has at most two nontrivial modules.

(ii) Let M_1, x, and y be as in case (i). As G is $\mathbf{\Delta}$-free and z is an odd extension with all neighbors in M_1, $M_1 + z$ satisfies the first two properties of extended halfgraphs. Suppose there are $x' \in X_1$ and $y' \in Y_1$ such that none of $zx', x'y', y'z$ is an edge. First observe that as $xy \notin E$, one of xy' and $x'y$ is also not an edge (M_1 does not contain $\mathbf{II}$); by symmetry we can assume that $x'y \notin E$. As we noted at the beginning of the proof, the graph induced by $C + x'$ is connected. Denote the neighbor of x' in C by y^*. As $x'y^*$ is an edge, y^* differs from z, y, y' and is in Y. zy^* is not an edge because C was induced. Therefore, $x'y^*, zy$ and y' induce $\mathbf{\Xi}$, a contradiction.

(iii) Let $z_1, z_2 \in Z$ be vertices with neighbors in the same module $M_1 = [X_1, Y_1]$. For proving $z_1 z_2 \notin E$ it is enough to see that they have a common neighbor in M_1. Let $y_1 \in Y_1$ be an isolated vertex of M_1. Either it is a common neighbor and we are done, or, e.g., $z_1 y_1 \notin E$. Then by (ii) and by the third property of extended halfgraphs

z_1 is connected to every vertex in X_1 and by the first property z_2 is connected to at least one vertex in X_1.

Let $z_1z_2 \notin E(G)$, let $x_i \in X, y_i \in Y$ be neighbors of z_i in M_i for $i = 1, 2$, and suppose $M_1 \neq M_2$. Observe that the subgraph D' induced by $\{z_1, x_1, y_2, z_2, x_2, y_1\}$ is a C_6. Since $\widehat{C_6} = 3K_2$, D' has three nontrivial modules. Hence, by (i), $D' \subset G$ has no odd extensions, which contradicts $M_1 \neq M_2$.

(iv) The first statement says that none of the bigraphs $[A, B], [B, C]$, and $[C, A]$ contains **II**. As we know this fact about $[X_1, Y_1]$ it is enough to prove it for $[Z, X_1]$. Suppose z_1x_1 and z_2x_2 induce a **II**. If X has a vertex connected to neither z_1 nor z_2, then G contains $\boldsymbol{\Xi}$. Hence $X = X_1$. Now if a vertex in Y_1 is isolated in M_1, then it is also isolated in D. Since the vertices z_i are odd extensions, both of them must have a neighbor in Y_1 which is not isolated in M_1. However, in the halfgraph M_1 there exists an $x \in X_1$ which is connected to every nonisolated vertex of Y_1, so x cannot be connected to any of the z_i's.

To prove the second statement indirectly, suppose that $a \in N_A(c_2) \setminus N_A(c_1)$ and $b \in N_B(c_2) \setminus N_B(c_1)$. Since ac_2b is not a triangle, $ab \notin E$. Since ac_1b is not an empty triangle, $ab \in E$. □

In the next proposition we formulate a converse of (2.6) which shows that a graph with properties (i)–(iv) is $\boldsymbol{\Xi}$-free. To get an even nicer symmetry, we swap the role of X_1 and Y_1. The proof is routine and the details are left to the reader.

(2.7) Suppose that the vertices of a graph G are partitioned into six nonempty sets $A_{i,j}, 1 \leq i \leq 3, 1 \leq j \leq 2$, such that the graph induced by A_{i_1,j_1} and A_{i_2,j_2} is a complete bipartite graph if $i_1 = i_2, j_1 \neq j_2$; a halfgraph or reduced halfgraph if $i_1 \neq i_2, j_1 = j_2$; and a graph with no edges otherwise. Suppose furthermore that for any $x \in A_{1,j}, y \in A_{2,j}, z \in A_{3,j}$ the set $\{x, y, z\}$ induces neither a triangle nor the complement of a triangle. Then G is $\boldsymbol{\Xi}$-free. □

3. On-line 3-coloring of $(\boldsymbol{\Delta}, \boldsymbol{\Xi})$-free graphs.

Let G be $(\boldsymbol{\Delta}, \boldsymbol{\Xi})$-free graph. If G is of Type 1, then by (2.2) and (2.3) it is either bipartite or 3-chromatic. Assume now that G is of Type 2 and nonbipartite. Then it has the structure described in (2.6). In particular, there is a bipartite subgraph $[X, Y]$ with nontrivial modules $M_1 = [X_1, Y_1]$ and $M_2 = [X_2, Y_2]$ such that all odd extensions can be partitioned into sets Z_1 and Z_2 in a manner that a vertex in Z_i has neighbors only in Z_{3-i} and in M_i. Since the three sets X, $Z_1 \cup (Y \setminus Y_1)$ and $Z_2 \cup Y_1$ are all independent, we get the following result:

(3.1) If G is a $(\boldsymbol{\Delta}, \boldsymbol{\Xi})$-free graph, then $\chi(G) \leq 3$. □

This section contains the proof of Theorem 1 (stated in the introduction), which claims that the stronger $\chi^*(G) \leq 3$ also holds in (3.1). Let us consider the on-line coloring game on graph G. At some step of the game let $D \subset G$ denote the colored subgraph (i.e., the subgraph induced by the set of all colored vertices of G), and denote by z the current vertex to be colored. For any on-line coloring algorithm A and for an integer r, let $A(r)$ denote the set of all vertices of G colored with r. If x is a colored vertex, $c(x)$ will denote its color.

Our on-line algorithm $\mathcal{A}$ consists of three consecutive stages. In the first stage, called *FF-stage*, first fit coloring is applied. The FF-stage ends up when **II** first appears in $D + z$. The current vertex z that terminates the FF-stage gets a color in the second stage including a single step, called **II**-*step*. After a suitable color is assigned to z a bigraph $D_0 \subseteq D + z$, called *reference graph*, is defined in the **II**-step to start the last stage. In the *last stage* z is considered as the (bipartite or odd) extension of the actual reference graph D_0. In each step of the last stage the color of z is determined with respect to D_0 and D_0 is updated for the next step.

FF-stage. z gets the smallest color r such that z has no neighbor in D colored with r.

Note that FF assigns the same color to equivalent vertices. In the early steps of the coloring game D is bipartite and eventually is disconnected. In this case we assume, for convenience, that all isolated vertices of $D = [X, Y]$ belong to Y. In particular, $X \neq \emptyset$ implies that $A(2) \cap X \neq \emptyset$. Notice, however, that the partite set of an isolated vertex is undefined in D, that is, it may change at a subsequent step of the game.

(3.2) If G is a $(\mathbf{\Delta},\mathbf{II})$-free graph, then $\chi_{FF}(G) \leq 3$.

Proof. If G is nonbipartite, then by (2.2) its primitive representative is C_5 or $C_5 + K_1$. Since, in both cases, the maximum degree is 2, $\chi_{FF}(G) \leq 3$ follows. Note that the coloring of C_5 by FF is unique: 12123 (in some cyclic ordering of the vertices).

Assume now that G is bipartite and contains at least one edge. Now G is a reduced halfgraph. Recall that all isolated vertices are considered to be in Y. In any FF-coloring of G, by definition, FF(1) is a maximal independent set, and FF(2) is a maximal independent set in $G - \text{FF}(1)$. So either FF(1)=Y and FF(2)=X or FF(1) is a maximal independent set containing vertices from both X and Y and FF(2)$\subseteq X$, FF(3)$\subseteq Y$ such that each 2-colored vertex is connected to every 3-colored vertex (because a reduced halfgraph minus a maximal independent set is either a graph with no edges or a complete bipartite graph). □

The properties of the coloring patterns obtained in the proof of (3.2) will be used in the **II**-step below.

II-step. We shall see that starting with this step $\mathcal{A}$ is able to color the current vertex z so that the overall colored graph satisfies a set of properties we call Ruleset.

Ruleset for a graph D. In D there is maximal induced bigraph $D_0 = [X, Y]$ with nontrivial modules $M_1 = [X_1, Y_1]$, $M_2 = [X_2, Y_2]$, etc., and with some trivial modules such that all odd extensions of D_0 are connected to either M_1 (forming the set Z_1) or M_2 (forming the set Z_2). Furthermore, the coloring by $\mathcal{A}$ satisfies the following rules:

For some permutation s_1, s_2, s_3 of colors 1, 2, and 3,

(i) if x and y are equivalent vertices in D, then $c(x) = c(y)$;
(ii) $A(s_3) \cap Y \subseteq Y_1 \subseteq Y \subseteq A(s_1) \cup A(s_3)$;
(iii) $A(s_3) \cap X \subseteq X_1 \subseteq X \subseteq A(s_2) \cup A(s_3)$;
(iv) $Z_2 \subseteq A(s_3)$, $Z_1 \subseteq A(s_1) \cup A(s_2)$;
(v) the bigraphs $[A(s_1) \cap Y_1, A(s_2) \cap X_1]$, $[A(s_2) \cap Z_1, A(s_3) \cap Y_1]$, and $[A(s_1) \cap Z_1, A(s_3) \cap X_1]$ are complete.

In the **II**-step $\mathcal{A}$ determines D_0 and properly colors z so that $D + z$ satisfies Ruleset. Note that as the colored graph extends, the same Ruleset will be maintained by $\mathcal{A}$ in each step of the last stage. Now we show how Ruleset can be achieved in the **II**-step.

Take any **II** in $D + z$ and let $D_0 = [X_0, Y_0]$ be a maximal bipartite subgraph of $D + z$ containing that **II**. Clearly z is in D_0. As $D_0 - z$ is **II**-free, D_0 has exactly two nontrivial modules $M = [X, Y]$ and $M' = [X', Y']$ and we can assume $\{z\} = X$. Let y be an arbitrary vertex in Y.

Case 1. D is bipartite. Only M can have odd extensions forming the set Z.

If the graph D is colored by two colors, then let $s_1 = c(y)$, $s_2 = 3 - s_1$, $s_3 = 3$, and color z by 3. If D has no isolated vertices, then D is connected so the color of a vertex is uniquely determined by its distance (in D) from y and Ruleset is satisfied. Suppose D has some isolated vertices, those isolated vertices are in Y'. (They are connected to z because $D + z$ is $\mathbf{\Xi}$-free and they cannot be odd extensions.) If $c(y) = s_1 = 1$, then Ruleset remains satisfied. If $c(y) = s_1 = 2$, then change s_2 to 3 and s_3 to 1. Now with $M_1 = M'$ Ruleset is satisfied again. ($X' \subseteq A(1) = A(s_3)$, $Z \subseteq A(1) = A(s_3)$ and there cannot be trivial modules in X_0.)

If D is 3-colored, then first suppose that if $t \in Y$, then $c(t) \neq 1$. Thus all vertices in Z have color 1. Let $s_1 = c(y)$, $s_2 = 5 - s_1$, $s_3 = 1$, and color z by s_2. Note that in $(D_0 - z) \cup Z$ there is a complete bipartite graph between the s_1- and the s_2-colored vertices. Consequently every s_2-colored vertex which is different from z is connected to y. As vertices in Z have color s_3 all vertices with color s_2 are in X_0. Thus color s_2 for z is permitted and all vertices with color s_1 are in Y_0. The isolated vertices of D are in Y' as before so if none of the trivial modules is colored by $s_3 = 1$, then Ruleset is satisfied with $M_1 = M'$. As D is 3-colored, color 1 appears in Y_0, so trivial modules in X_0 have different color. If $y^* \in Y_0$ is a trivial module colored by 1, then X' is uniformly colored by s_2. In this case vertices in $Y_0 \setminus Y$ have no neighbors colored by 1, so they themselves are colored by $1 = s_3$ and consequently swapping s_1 and s_3 and choosing $M_1 = M$ Ruleset is satisfied again.

Now suppose D is 3-colored and there is a $t \in Y, c(t) = 1$. Observe that Y cannot be uniformly 1-colored because in this case color 3 could not appear in D, so there is a $t' \in Y, c(t') \neq 1$. Let $s_1 = 1$, $s_3 = c(t')$, $s_2 = 5 - s_3$. Now vertices in $X_0 - z$ are colored by s_2, so vertices in $Y_0 \setminus Y$ are colored by $1 = s_1$ and there is no s_2-colored vertex in Y. The s_3-colored t' has a 1-colored neighbor z_1; it must be in Z. If $z_2 \in Z$ is colored by s_3, then $z_1 t'$ and $z_2 t$ induce a $\mathbf{II}$, which is not the case. Thus we are allowed to color z by s_3. To check that Ruleset is satisfied with $M_1 = M$ we need to check rule (v). Every vertex in Z is connected to z. Let $z_2 \in Z$ be an arbitrary vertex colored by s_2. By (2.6)(iv) one of $N_Y(z_1)$ and $N_Y(z_2)$ contains the other. Clearly $N_Y(z_1) \subset N_Y(z_2)$ because z_1 does not have any 1-colored neighbor in Y while z_2 does. Therefore, the arbitrarily chosen s_2-colored $z_2 \in Z$ and s_3-colored $t' \in Y$ are connected.

Case 2. D is not bipartite. As $D+z$ is of Type 2, it is connected by the observation made at the beginning of the proof of (2.6)—every nonbipartite subgraph must be connected. Thus D is a vertex multiplication of C_5.

Let Z be the (maybe empty) set of odd extensions connected to M and Z' be the set of odd extensions connected to M'. As D is a vertex multiplication of C_5 both M and M' are vertex multiplications of H_1 and either Z is empty or there are no trivial modules in Y_0. Moreover, the equivalence classes of the C_5 are uniformly colored and one class is colored by 3 while the others are colored by 1212. It is easy to check (five cases depending on which class is colored by 3) that we can color z in a manner such that every $\mathbf{II}$ in $D + z$ will be colored by three colors and Ruleset is satisfied in all these cases with appropriate permutation s_1, s_2, s_3. □

Last stage. When the algorithm observes that D contains $\mathbf{II}$ it knows that D satisfies Ruleset and is able to determine appropriate D_0, M_1, M_2, and permutation s_1, s_2, s_3. In every step of the last stage $\mathcal{A}$ colors z in such a way that $D + z$ always satisfies Ruleset. In particular, G becomes 3-colored when $\mathcal{A}$ terminates.

Case 1. D is connected and z is an odd extension of D_0. If z is connected to M_2, then color z by s_3 and $D + z$ clearly satisfies Ruleset. Furthermore, suppose z is connected to M_1. If all neighbors of z are colored by s_3, then z is uniformly connected to either $A(s_3) \cap X_1$ or $A(s_3) \cap Y_1$; otherwise an empty triangle could be found. In the first case s_1 and in the second s_2 is the appropriate color for z to satisfy rule (v). Now by symmetry we can suppose that z is connected to an $x \in X_1$ such that $c(x) = s_2$. The following line of thought will be used in further cases:

(3.3) We claim that if $x' \in X_1$ and $c(x') = s_3$, then $zx' \in E$. For getting a contradiction suppose that z is connected to x but not to x'. By this assumption and by (2.6)(iv) $N_{Z_1+z}(x) \supset N_{Z_1+z}(x')$ in graph $D + z$ and so $N_{Y_1}(x) \subseteq N_{Y_1}(x')$. In the graph D all vertices in Z and Y_1 have a color satisfying rule (v), so $N_{Z_1}(x) = N_{Z_1}(x')$ and $N_{Y_1}(x) = N_{Y_1}(x')$. We get $x \sim x'$ in D and this contradicts rule (i).

To finish Case 1 observe that if z is connected to any $y \in Y_1$, $c(y) = s_1$, then xyz would be a triangle. The argument above says that z is connected to every s_3-colored vertex in X_1; consequently s_1 is the proper color for z.

Case 2. D is connected and z is a bipartite extension of D_0. Let $M = [X, Y]$ be the module of $D_0 + z$ which contains z. By symmetry we can assume $z \in X$. If $M \not\supseteq M_1$, then color z by s_2 and Ruleset remains satisfied. Suppose $M_1 \subseteq M$. First observe that vertices in $M \setminus M_1$ different from z cannot make any rule wrong. (They are trivial modules of D_0.) If z is equivalent to some vertex in M, then the color of that vertex is also good for z.

Suppose there is a $y \in Y$, $c(y) = s_3$, and $zy \in E$. If $z_1 \in Z_1$ and z is connected to z_1, then y is not connected to z_1 so $c(z_1) \neq s_2$. A similar argument as in (3.3) shows that z must be connected to every s_1-colored vertex in Y; consequently s_2 is a proper color for z.

The remaining case is that all neighbors of z in Y are colored by s_1. If z is uniformly connected to $A(s_1) \cap Z_1$, then s_3 is a proper color for z. If there is a $z_1 \in Z_1$ such that $c(z_1) = s_1$ and zz_1 is not an edge, then the absence of empty triangles shows that every s_1-colored vertex in Y is connected to z. An argument similar to (3.3) says that z has no s_2-colored neighbors in Z_1, so s_2 is the proper color for z.

Case 3. D is not connected. Now D is a vertex multiplication of **II** and is 3-colored (satisfying Ruleset). Note that when adding z to D some vertices might change their partite sets. To resolve this problem consider $D + z$ that is a connected bigraph (with a unique bipartition). Then remove z and keep the eventually modified bipartition for D. It is easy to check that in each case a permutation s_1, s_2, s_3 can be obtained so that Ruleset holds true for the modified bigraph D. Then the procedure described in Case 2 applies. This concludes the proof of Theorem 1.

4. $\boldsymbol{\Delta}$-free critical graphs of on-line chromatic number 4. In this section we characterize $\boldsymbol{\Delta}$-free 4-critical graphs. Obviously, every graph G with on-line chromatic number 4 must contain an induced subgraph G' such that $\chi_{FF}(G') = 4$. In [GKL1] we list all graphs of FF-chromatic number 4 which are minimal. From the list of these 22 graphs the $\boldsymbol{\Delta}$-free ones are F_i, $1 \leq i \leq 5$, shown in Figure 2. It is also shown in [GKL1] that F_2, F_3, and F_4 are 4-critical graphs, $F_1 = \mathbf{B}$ and F_5 are not. This is formulated in the following proposition.

(4.1) Let G be a $\boldsymbol{\Delta}$-free 4-critical graph. Then either G is isomorphic to one of F_2, F_3, and F_4 or G contains at least one of $\mathbf{B}$ and F_5, shown in Figure 2. □

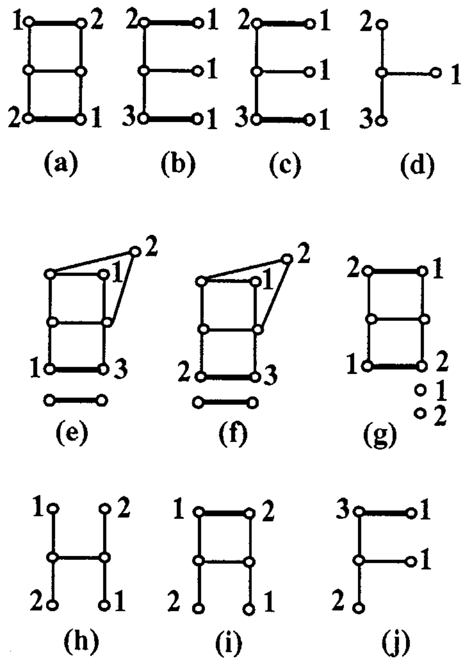

Fig. 5.

The list of all 4-critical graphs containing F_5 is obtained in [GKL1] and shown in Figure 3. The analysis performed in this section results in a list of 4-critical graphs containing a copy of **B**; see Figure 4. First we show that every graph in Figure 4 has on-line chromatic number 4. Then we prove that the list contains 4-critical graphs and is complete. We discuss connected and disconnected graphs separately in sections 4.1 and 4.2.

To prove $\chi^*(B_i) \geq 4$ we show that Drawer has a 4-forcing strategy against Painter for every $1 \leq i \leq 10$. Let $v_1, v_2, \dots$ be the order of vertices of G as revealed by Drawer, and let D_k be the colored subgraph after the kth step of the coloring game.

BE-strategy. Let D_4 be isomorphic to **II**. If D_4 becomes 2-colored, then Drawer wins on **B** (see Figure 5(a)). If D_4 is 3-colored, say, $(1,2)$ and $(1,3)$ are the colored edges, then let v_5 be an isolated vertex. Painter essentially has two different choices to color v_5. In both cases Drawer wins on **E** (see Figures 5(b) and 5(c)). It is easy to check that $\mathbf{E} \subset B_i$ for $1 \leq i \leq 4$; thus $\chi^*(B_i) \geq 4$ is satisfied by these graphs.

Pigeonhole strategies. Let v_1, v_2, v_3, v_4 be isolated vertices. If D_4 contains three vertices of the same color, say, v_1, v_2, v_3 are colored 1, then Drawer reveals v_5, v_6, v_7 with edges v_1v_5, v_2v_6, and v_3v_7. In D_7 two of these edges have the same coloring pattern, say, $(1,2)$, and Drawer wins on **B** (see Figure 5(a)). If D_4 contains three vertices of distinct colors, then Drawer wins on a "claw" (see Figure 5(d)). This strategy is feasible if the graph has $3K_2 + K_1$. Therefore, one may assume that for every B_i, $i = 7, 8$, and 9, D_4 is 2-colored according to the pattern $(1,1,2,2)$. From the fifth step the strategy depends on the graph in question. For B_7, a fifth isolated vertex v_5 is Drawer's winning move. Indeed, by this move Drawer forces three vertices of the same color or three distinctly colored vertices; both are winning positions for Drawer as before. For B_8, the winning position is $4K_2$. In that case D_8 always has two edges

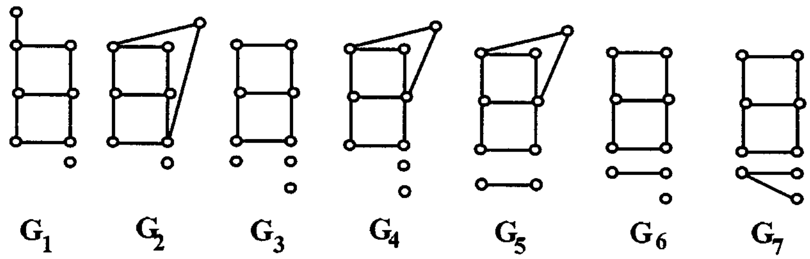

FIG. 6.

with the same coloring pattern, by the pigeonhole principle, and then Drawer wins on **B**.

Ξ-variant. In the case of B_9 and B_{10} Drawer starts with the pigeonhole strategy and obtains three isolated vertices v_1, v_2, v_3 with coloring pattern $(1, 1, 2)$. Then v_4 is given by Drawer with edge v_1v_4 and a new edge, v_5v_6 (so until now a **Ξ** is given). If Painter uses color 3, then Drawer wins as in Figure 5(e) or as in Figure 5(f); otherwise the edges are colored by 1 and 2 and Drawer wins as in Figure 5(g).

A-variant. The strategy for B_5 and B_6 differs from the fourth step; however, its elements are the same as before. After the first three isolated vertices D_3 contains two vertices with the same color, say, v_1, v_2 are colored 1 and v_3 is colored 2. Then Drawer's winning move consists in giving v_4 with edge v_1v_4. Depending on the color of v_4 (2 or 3) Drawer wins on graph **A** or on **F** (see Figures 5(i) and 5(j)).

In the next step of our analysis we show that all graphs in Figure 4 are 4-critical. The removal of any vertex of B_i, $1 \leq i \leq 4$, results in a ($\boldsymbol{\Delta}$-free) graph which is either **Ξ**-free or **B**- and F_5-free. In the first case the proper subgraphs have on-line chromatic number at most 3, by Theorem 1. In the second case FF is obviously a 3-coloring (c.f. (4.1)). Hence B_i, is 4-critical for $1 \leq i \leq 4$. To see that B_i is 4-critical for $5 \leq i \leq 10$ it is enough to check the on-line chromatic number of its proper subgraphs containing **B** (otherwise FF is a 3-coloring). Among all of these graphs it is enough to deal with the maximal ones: G_j, $1 \leq j \leq 7$, listed in Figure 6.

Since algorithm $\mathcal{A}$ defined in section 3 works also for ($\boldsymbol{\Delta}$-free) graphs with a **Ξ**-free connected component plus any number of isolated vertices, $\chi_{\mathcal{A}}(G_j) \leq 3$ follows for $1 \leq j \leq 4$. The on-line 3-colorability of G_5, G_6, and G_7 will be settled in section 4.2.

4.1. Connected 4-critical graphs. Let G be a connected $\boldsymbol{\Delta}$-free graph containing **B**. The main goal of the present section consists of proving Theorem 2, which states that the following statements are equivalent:

(1) G is **E**-free.
(2) G has no induced subgraph isomorphic to any of F_3, F_4 in Figure 2 and B_1, B_2, B_3, B_4 in Figure 4.
(3) G is **Ξ**-free.
(4) G has on-line chromatic number $\chi^*(G) \leq 3$.

Our algorithm $\mathcal{A}$ in the proof of Theorem 1 is an on-line 3-coloring whenever G is **Ξ**-free; thus we have (3) $\Longrightarrow$ (4). If G contains both **B** and **E**, then Drawer may use the **BE**-strategy mentioned above and forces a 4-coloring; hence (4) $\Longrightarrow$ (1). Observe that all graphs in (2) contain a copy of **E**, thus (1) $\Longrightarrow$ (2) . Therefore, it is enough to prove the remaining implication (2) $\Longrightarrow$ (3).

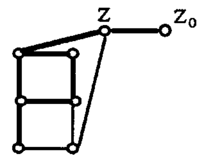

Fig. 7.

(4.2) Let G be a connected $\mathbf{\Delta}$-free graph containing $\mathbf{B}$. Then G is $\mathbf{\Xi}$-free if and only if G has no induced subgraph isomorphic to any of F_3, F_4 in Figure 2 and B_1, B_2, B_3, B_4 in Figure 4.

Proof. Since all forbidden graphs contain $\mathbf{\Xi}$, necessity is obvious. We prove sufficiency by contradiction. Suppose there exists a minimal counterexample G containing $\mathbf{\Xi}$. Let D be a maximal bipartite $\mathbf{\Xi}$-free subgraph of G such that it contains a copy of $\mathbf{B}$. First we show that every vertex of $V(G) \setminus V(D)$ has a neighbor in D. Suppose to the contrary that there are vertices $z, z_0 \notin V(D)$ such that zz_0 is an edge, z_0 has no neighbors in D, and $D + z$ is connected. By the minimality of G and by the choice of D, it follows easily that $D = \mathbf{B}$ and z is an odd extension of $\mathbf{B}$. Consequently, $G = (\mathbf{B} + z) + z_0$ is the graph shown in Figure 7, which contains F_3, a contradiction. Hence $D + z$ is connected for every $z \in V(G) \setminus V(D)$.

The proof of (4.2) (i.e., that the counterexample G does not exist) is arranged in three steps. Let z be called an *illegal extension* of D if $D + z$ contains $\mathbf{\Xi}$. In Steps 1 and 2, we show that D has no illegal (bipartite or odd) extension. In Step 3 we prove that the set of all odd extensions of D satisfy the conditions required by the structure theorems in section 2. The contradiction is obtained by (2.7), which implies that G is $\mathbf{\Xi}$-free.

Step 1. We show that the bigraph $D = [X, Y]$ has no illegal bipartite extension. Suppose on the contrary that $z \in V(G) \setminus V(D)$ is an illegal bipartite extension of D. By symmetry, one may assume that z extends X, which is adjacent to some vertex of Y. Note also that z is nonadjacent to some vertex of Y (since otherwise it would not be illegal). To get a contradiction, we shall show that $D+z$ contains one of F_3, B_1, B_2, and B_3 or, equivalently, the bipartite complement $\widehat{D+z}$ contains one of $\widehat{F_3} = P_6 + K_1$, $\widehat{B_1} = P_5 + 2K_1$, $\widehat{B_2} = F_2$ (see Figure 2), and $\widehat{B_3} = \mathbf{E} + K_2$. For convenience, we are working on the bipartite complement of G, and $G^* = \widehat{D+z}$ is considered as the extension of $\widehat{D}$. Note that Y contains both neighbors and nonneighbors of z also in G^*. Let $G_i = [X_i, Y_i]$ be the nontrivial connected components of $\widehat{D}$, $1 \le i \le k$. By (2.4), each G_i is a connected reduced halfgraph. Since D contains $\mathbf{B}$, and since $\widehat{\mathbf{B}} = \mathbf{\Xi}$, we have $k \ge 2$. From the assumption that z is an illegal extension it follows that G^* has a P_5.

Assume that G_1 has a pair of nonadjacent vertices $x \in X_1$, $y \in Y_1$. Supposing that z is (uniformly) nonadjacent to Y_1 any P_5 avoids G_1 and together with $\{x, y\}$ induces a $P_5 + 2K_1 \subset G^*$. Suppose now that z is uniformly adjacent to Y_1, and consider a P_5 induced by $\{x_1, y, z, y_2, x_2\}$, where $x_1 \in X_1$, $x_2 \in X_2$, and $y_2 \in Y_2$ (a P_5 in this form must exist). Then some $y' \notin Y_1$ is nonadjacent to z. Hence $\{x_1, y, z, y_2, x_2, y', x\}$ induces a $P_6 + K_1$ or $P_5 + 2K_1$ in G^*, depending on whether $x_2 y'$ is an edge (see Figure 8(a)). As a corollary, one may assume that for each G_i $(1 \le i \le k)$ different from the complete bigraph, $G_i + z$ contains one of L_1 and L_2 in Figure 8 as an induced subgraph. (z has both neighbors and nonneighbors in Y_i and $[X_i, Y_i]$ is a connected reduced halfgraph.)

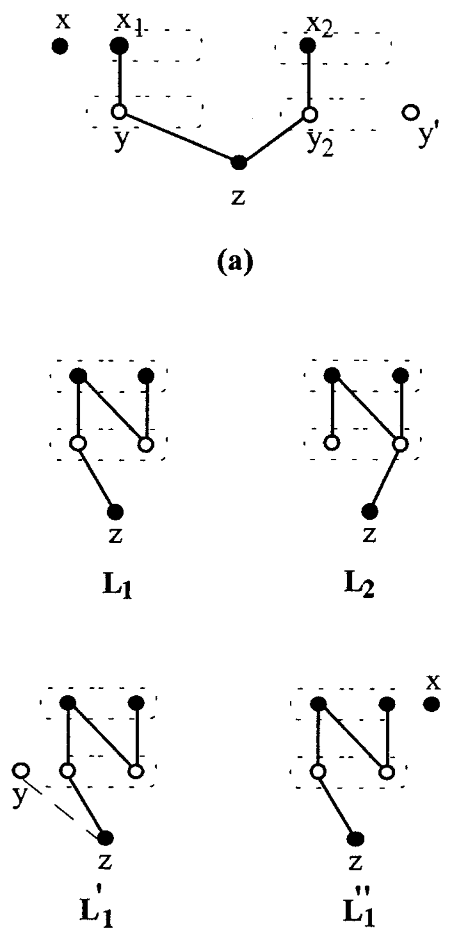

FIG. 8.

If there exist two components different from the complete bigraph, then G^* contains the union of L_i and L_j $(1 \leq i \leq j \leq 2)$ sharing a common vertex z. Since this union contains a $P_6 + P_1$ for each of the three possible choices of (i, j), one may assume that all but possibly one component is a complete bigraph.

Case a. G_1 is not a complete bigraph.

First suppose that $L_1 \subset G_1 + z$. To get a contradiction we show that there exists a copy of L_1 and there are two nonadjacent vertices $x \in X$, $y \in Y$ not in L_1 such that x and y have no neighbor in $L_1 - z$. Since $k \geq 2$ and G_2 contains an edge x_2y_2, the claim follows if G^* has at least one more component. If this is not true, then (since $\mathbf{B} \subset D$) it follows easily that $G_1 + z$ contains either L_1' or L_1'' in Figure 8 with x or y in G_1. Obviously, L_1 and x_2y_2 together with x or y contain either a $P_5 + 2K_1$ or a $P_6 + K_1$.

Suppose now that $L_2 \subset G_1 + z$. If $G_1 + z$ has a P_5 (in this case it must have an L_1 as well), then we are done by using the previous argument. Hence there exists an edge zy, for some y not in G_1. Let $x_2y_2 \in E(G_2)$. If x_2y is a nonedge, then $V(L_2) \cup \{x_2, y_2, y\}$ induces either an $\mathbf{E} + K_2$ or an F_2, depending on whether z and y_2 are nonadjacent or adjacent (see Figure 9(a)). If $y \neq y_2$, zy_2 is a nonedge and x_2y

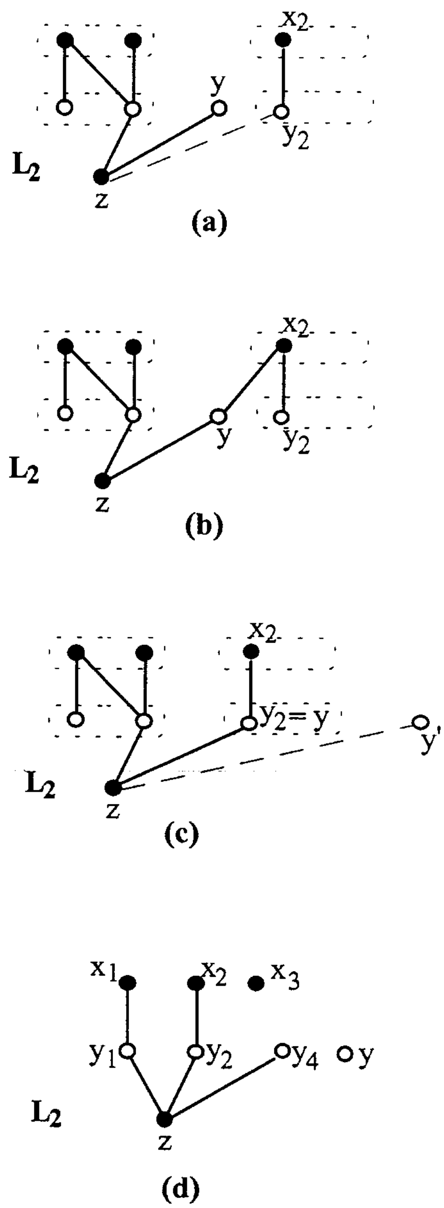

FIG. 9.

is an edge, then the subgraph induced by $V(L_2) \cup \{x_2, y_2, y\}$ contains $P_6 + K_1$ (see Figure 9(b)). Hence one may assume that $y = y_2$. If some $y' \notin Y_1$ is nonadjacent to x_2, then the subgraph induced by $V(L_2) \cup \{x_2, y_2, y'\}$ either contains a $P_6 + K_1$ or induces an F_2 (see Figure 9(c)). If this last condition does not hold, then (using $\mathbf{B} \subset D$) we easily obtain the existence of $y' \in Y_1$ nonadjacent to L_2. Then the subgraph induced by $V(L_2) \cup \{x_2, y_2, y'\}$ contains a $P_6 + K_1$.

Case b. G_i is a complete bigraph for every $i = 1, \ldots, k$.

Suppose that (x_1, y_1, z, y_2, x_2) is a P_5 with $x_i \in X_i$ and $y_i \in Y_i$ $(i = 1, 2)$. Since $\mathbf{B} \subset D$, there are two more components (possibly trivial) containing vertices $x_3 \in X$ and $y_4 \in Y$. One may assume that $zy_4 \in E(G^*)$ (because otherwise a $P_5 + 2K_1$ is found). Then by the connectivity of $D + z$, we have $yz \notin E(G^*)$ for some $y \in Y$. If yx_3 is a nonedge, then we get a $P_5 + 2K_1$; otherwise, $\{x_1, y_1, z, y_2, x_2, y_4, y, x_3\}$ induces an $\mathbf{E} + K_2$ (see Figure 9(d)).

In each case there is a forbidden configuration; therefore, D has no illegal bipartite extension.

Step 2. Next we show that D has no illegal odd extension. Suppose to the contrary that $z \in V(G) \setminus V(D)$ is an illegal odd extension of the bigraph $D = [X, Y]$. Consider the modular decomposition of D and let $M_i = [X_i, Y_i]$, $1 \leq i \leq k$, be the nontrivial modules. Since D contains a $\mathbf{B}$, we have $k \geq 2$. Denote by $X(z) \subset X$ and $Y(z) \subset Y$ the set of all neighbors of z in X and Y, respectively. Since z is an odd extension of D and G is $\mathbf{\Delta}$-free, $X(z)$ and $Y(z)$ are nonempty sets belonging to the same module, say, M_1, and $X(z) \cup Y(z)$ is an independent set of D. Clearly, $G_X = (D + z) - X(z)$ and $G_Y = (D + z) - Y(z)$ are bipartite proper subgraphs of G. Moreover, one of them contains $\mathbf{\Xi}$ (since z is an illegal extension). Hence, by the minimality of G, $\mathbf{\Xi} \subset G_X$ (or $\mathbf{\Xi} \subset G_Y$) implies that G_X (or G_Y) either is disconnected or $\mathbf{B}$-free. This observation implies that $k = 2$ as follows. If $k \geq 3$, then G_X (and G_Y) is connected, and any $\mathbf{II}$ between M_2 and M_3 together with z would induce $\mathbf{\Xi}$ in G_X (and G_Y). Consequently, G_X (and G_Y) must be $\mathbf{B}$-free. In particular, $k \leq 3$ and D has no trivial module. Suppose that $k = 3$. If M_1 has an edge xy, then one of its end vertices is not in $X(z) \cup Y(z)$, say, $x \notin X(z)$. Then clearly G_X has a $\mathbf{B}$, which is not allowed. Hence M_1 has no edge. If one of M_2 and M_3 has an edge, then G_X (and G_Y) also contains a $\mathbf{B}$. We have obtained that D has exactly three modules, none of which contains an edge. Then D has no $\mathbf{B}$, a contradiction. Therefore, $k = 2$ follows.

Next we show that $M_1 + z$ is an extended halfgraph. Suppose that $H(n)$ is a primitive representative of M_1 with partite sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ with $x_i y_j$ an edge if and only if $i < j$. It is enough to prove that at least one of $\mathcal{C}_{M_1}(x_t)$ and $\mathcal{C}_{M_1}(y_t)$ is uniformly adjacent to z for every $1 \leq t \leq n$. First assume that there exist vertices $x \in \mathcal{C}_{M_1}(x_t) \cap X(z)$, $\tilde{x} \in \mathcal{C}_{M_1}(x_t) \setminus X(z)$, and suppose on the contrary that $\tilde{y} \in \mathcal{C}_{M_1}(y_t) \setminus Y(z)$. (By the symmetry of $\mathcal{C}_{M_1}(x_t)$ and $\mathcal{C}_{M_1}(y_t)$, our argument also applies when the roles of X and Y are interchanged.)

Let $x' \in X_2$ and $y' \in Y_2$ be nonadjacent vertices of $D - M_1$. If $t < n$, then let us choose an arbitrary vertex $y'' \in \mathcal{C}_{M_1}(y_n)$. Since $x \in X(z)$ and $xy'' \in E(D)$, we have $y'' \notin Y(z)$. Then the set $\{z, x, \tilde{x}, \tilde{y}, x'', y'', y'\}$ induces an F_3 (see Figure 10(a)), a contradiction. The same contradiction can be deduced for $t = n$ if there exists a vertex $y'' \in Y \setminus Y_1$ adjacent to x'.

We analyze further the case $t = n$ assuming that $D - M_1$ has no $\mathbf{\Xi}$ with its isolated vertex in Y. Then, from the condition $\mathbf{B} \subset D$, it follows easily that D contains a copy of $\mathbf{B}$ such that $x'y_i$, $y'x_j$ $(1 \leq i \leq j \leq n)$ are the top and bottom edges (i.e., edges of the $\mathbf{II}$-part) and x^*y^* with $x^* \in X$, $y^* \in Y_1$ is the middle edge (i.e., the edge between the two star vertices). Observe that any vertex of $\mathcal{C}_{M_1}(y_n)$ may play the role of y^*; thus (in the present case) we may set $y^* = \tilde{y}$. If $x_j \in X(z)$, then we get an F_3 (see Figure 10(b)). Thus we assume that $x_j \notin X(z)$ holds. Supposing that $x^* \notin X_1$, we may choose for y_i any vertex of Y_1 adjacent to z. This would result in a copy of F_3 (see Figure 10(c)); thus we may assume that $x^* \in X_1$ holds.

Supposing that $x^* \in X(z)$ (and because $x_j \notin X(z)$), we obtain a B_1 (see Figure 10(d)). Thus we may assume that $x^* \notin X(z)$ also holds. Regardless of whether zy_i

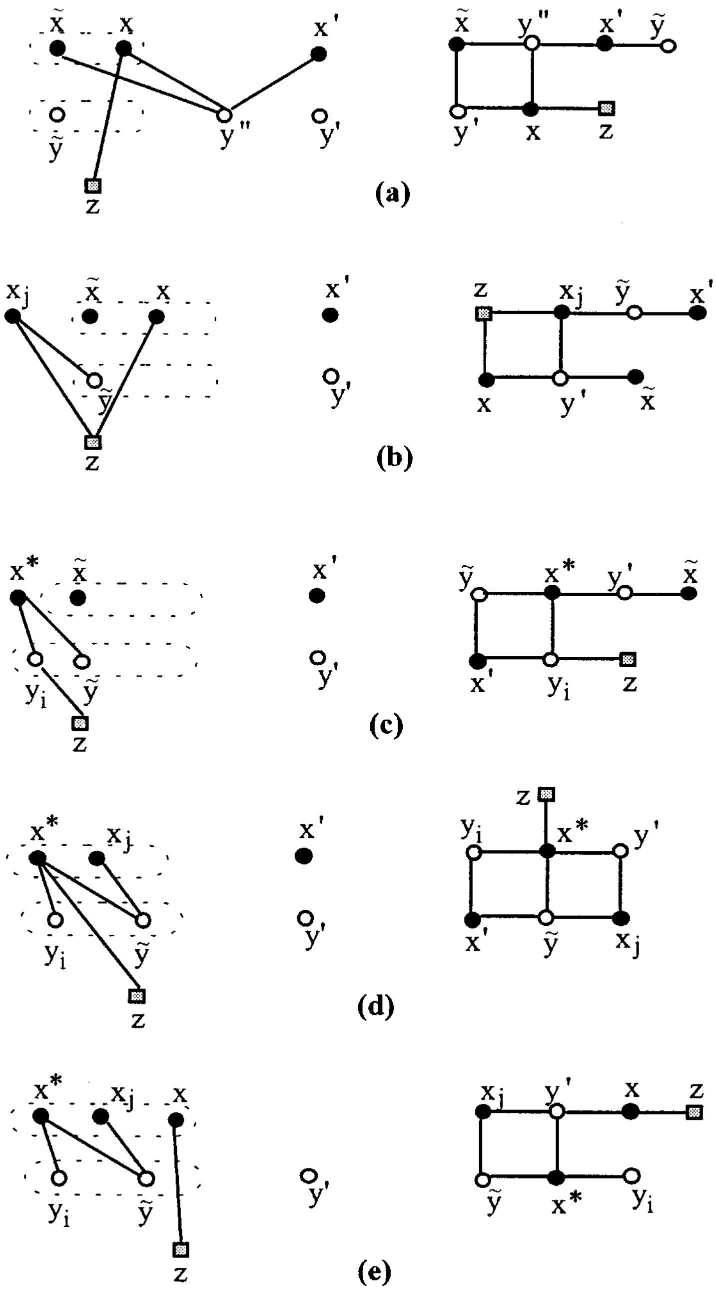

FIG. 10.

is an edge or a nonedge, we get an F_3; see Figure 10(e) if $y_i \notin Y(z)$ and Figure 10(c) otherwise. Thus we have obtained that the existence of the vertices $x \in \mathcal{C}_{M_1}(x_t) \cap X(z)$ and $\tilde{x} \in \mathcal{C}_{M_1}(x_t) \setminus X(z)$ $(1 \le t \le n)$ implies that $\mathcal{C}_{M_1}(y_t) \subset Y(z)$. Recall that the same is true when interchanging the role of X and Y.

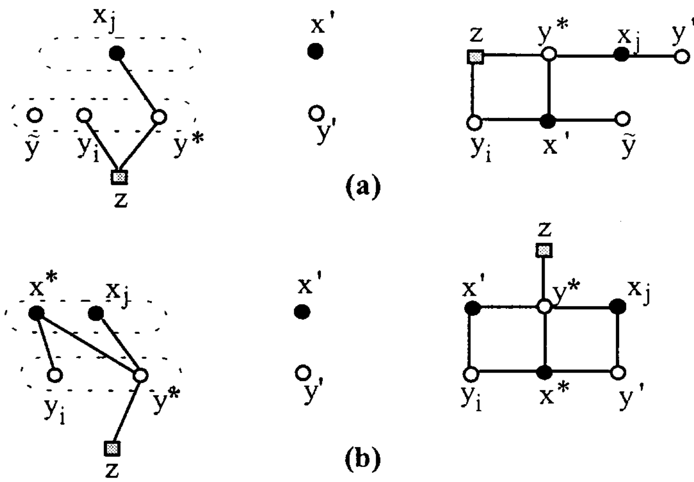

Fig. 11.

In the next step we show that $\mathcal{C}_{M_1}(y_1) \cap Y(z) \neq \emptyset$. Suppose that $\tilde{y} \notin Y(z))$ holds for every $\tilde{y} \in \mathcal{C}_{M_1}(y_1)$. Notice that $Y(z) \neq \emptyset$ implies that $n \geq 2$. Therefore (since G is $\mathbf{\Delta}$-free), there exist vertices $\tilde{x} \in \mathcal{C}_{M_1}(x_1) \setminus X(z)$ and $x \in X(z)$. If there is a vertex $y'' \in Y \setminus Y_1$ adjacent to x', then we get F_3 (see Figure 10(a)). Otherwise, consider again the copy of $\mathbf{B}$ with vertex set $\{x_j, y_i, x^*, y^*, x_j, x', y'\}$, where $y^* \in \mathcal{C}_{M_1}(y_n)$ and $x^* \in \mathcal{C}_{M_1}(x_1) \cup (X \setminus X_1)$. Notice that $x^* \notin X(z)$ holds (by the same argument as before).

If $x_j \in X(z)$, then we get F_3 as in Figure 10(a), with $x = x_j$, $\tilde{x} = x^*$, and $y'' = y^*$. Thus we may assume that $x_j \notin X(z)$ holds for every vertex in the role of x_j. Therefore (since $X(z) \neq \emptyset$), there exists $x \in \mathcal{C}_{M_1}(x_n) \cap X(z)$. If the situation is different from the one in Figure 10(e) (with $\tilde{y} = y^*$), then either $y_i \in Y(z)$ or $y^* \in Y(z)$ holds, but not both, since in this case we get an F_3 (see Figure 11(a)).

In either case we get an F_3: see Figure 10(c) (with $\tilde{y} = y^*$) if $y_i \in Y(z)$, and see Figure 11(b) if $y^* \in Y(z)$.

This proves that $\mathcal{C}_{M_1}(y_1) \cap Y(z) \neq \emptyset$. By the symmetry of halfgraphs, the same argument shows that $\mathcal{C}_{M_1}(x_n) \cap X(z) \neq \emptyset$. From the previous steps of the proof it follows that at least one of the properties $\mathcal{C}_{M_1}(y_t) \subset Y(z)$ and $\mathcal{C}_{M_1}(x_t) \subset X(z)$ holds for $t = 1$ and n.

To conclude the proof, suppose that there exist vertices $\tilde{x} \in \mathcal{C}_{M_1}(x_t) \setminus X(z)$ and $\tilde{y} \in \mathcal{C}_{M_1}(y_t) \setminus Y(z)$, for some $1 < t < n$. Let $y^* \in \mathcal{C}_{M_1}(y_n)$ and $x^* \in \mathcal{C}_{M_1}(x_1)$. If $y^* \in Y(z)$, then we get F_3 as in Figure 11(b) (with $x_j = \tilde{x}$ and $y_i = \tilde{y}$). If $x^* \in X(z)$, then we get F_3 as in Figure 10(c) (with $x_j = \tilde{x}$, $y_i = \tilde{y}$ and $\tilde{y} = y^*$). Assuming that $x^* \notin X(z)$, $y^* \notin Y(z)$, and choosing a vertex $x \in \mathcal{C}_{M_1}(x_n) \cap X(z)$, we get F_3 as in Figure 10(e) (with $x_j = \tilde{x}$, $y_i = \tilde{y}$, and $\tilde{y} = y^*$).

Hence, for every $1 \leq t \leq n$, at least one of $\mathcal{C}_{M_1}(x_t)$ and $\mathcal{C}_{M_1}(y_t)$ is uniformly adjacent to z. This implies that $M_1 + z$ is an extended halfgraph. In particular, by (2.7), $G + z$ is $\mathbf{\Xi}$-free, a contradiction. Therefore, D has no illegal odd extension.

Step 3. If the subgraph $G - z$ is bipartite for some $z \in V(G)$ and contains $\mathbf{B}$, then by the choice of D, $D = G - z$; furthermore, z is an illegal extension of D. This is not possible as we have seen in Steps 1 and 2. Therefore, by the maximality of D, one may assume that $G - D$ has at least two vertices, and each $z \in V(G - D)$ is a

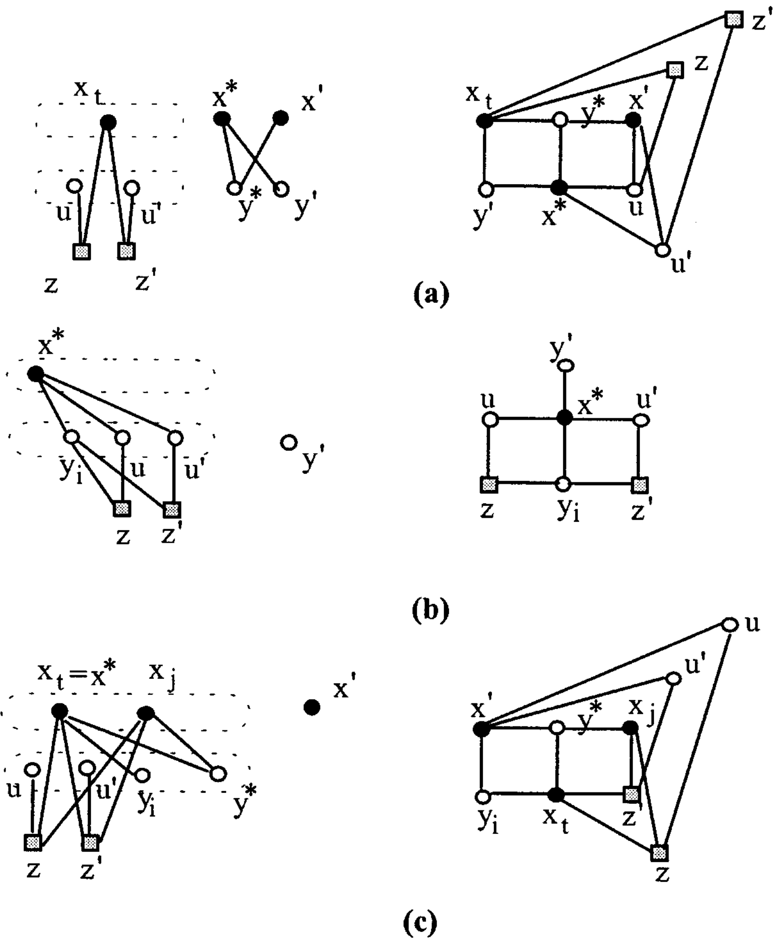

FIG. 12.

legal odd extension of D. Hence, by the structure result in section 2, D has exactly two nontrivial modules.

Let Z_i be the set of all odd extensions of D adjacent to M_i, $i = 1, 2$. By (2.6), for every $z \in Z_i$, $M_i + z$ form an extended halfgraph. By Step 2, and since G is $\mathbf{\Delta}$-free, Z_1 and Z_2 are independent sets.

First we prove that, for $Z_1 \neq \emptyset$, the bigraph $H = [Z_1, Y_1]$ is $\mathbf{II}$-free. Then, by symmetry, the same is true for the bigraphs $[Z_i, Y_i]$, $[Z_i, X_i]$ $(i = 1, 2)$. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be the partite sets of a primitive representative $H(n)$ of M_1 with $x_i y_j \in E(D)$ if and only if $i < j$. For any $z \in Z_1$, let $X(z) = \{x \in X_1 \ : \ xz \in E(G)\}$ and $Y(z) = \{y \in Y_1 \ : \ yz \in E(G)\}$. Assume that zu and $z'u'$ are edges of a $\mathbf{II} \subset H$, where $z, z' \in Z_1$ and $u, u' \in Y_1$. If u and u' are not equivalent in M_1, then there exists a vertex $x \in X_1$ such that, say, ux is an edge but $u'x$ is not. But if zx is an edge, then zux is a triangle; if it is not, then $zu'x$ is an empty triangle. Thus u and u' belong to the same equivalence class, $\mathcal{C}_{M_1}(y_t)$, for some $1 \leq t \leq n$.

Let $x' \in X \setminus X_1$ and $y' \in X \setminus X_1$ be nonadjacent vertices of $D - M_1$ and choose a copy of $\mathbf{B}$ such that $x'y_i$ and $y'x_j$ $(1 \leq i \leq j \leq n)$ are the top and bottom edges and

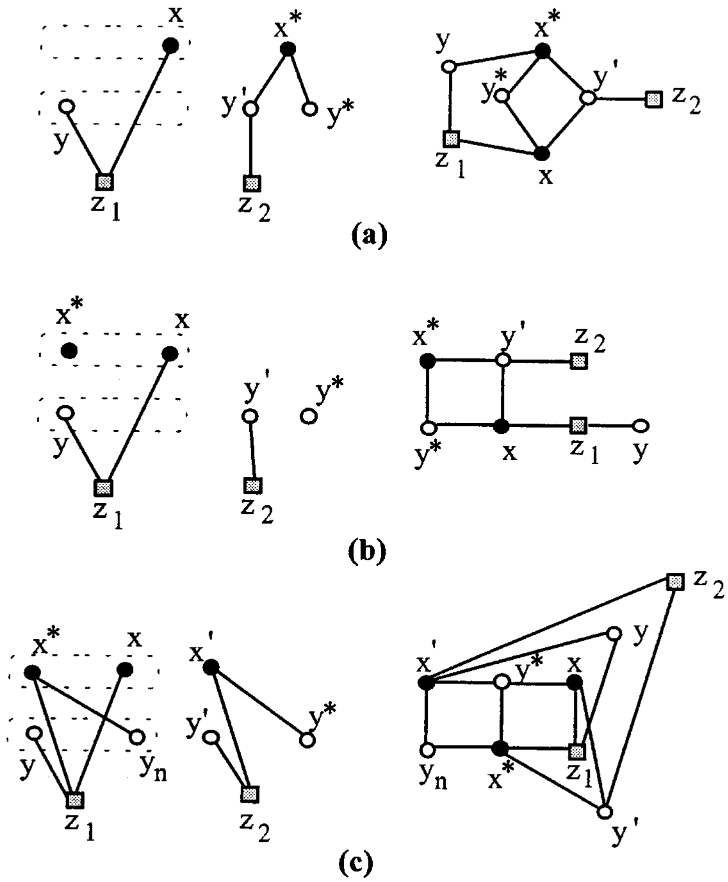

FIG. 13.

x^*y^* with $x^* \in X$, $y^* \in Y$ is the middle edge. If $x^* \in X \setminus X_1$ and $y^* \in Y \setminus Y_1$, then (since $x_t \in X(z) \cap X(z')$) we get a copy of B_4 (see Figure 12(a)).

One may assume that at least one of x^* and y^* is in M_1. This implies that $n \geq 2$; furthermore, every vertex of $\mathcal{C}_{M_1}(y_n)$ or $\mathcal{C}_{M_1}(x_1)$ may play the role of $x^* \in X_1$ or $y^* \in Y_1$. If $1 < t < n$, we obtain a B_4 (see Figure 12(a) with $x^* = x_1$, $y^* = y_n$). If $t = n$, then either $y^* \notin Y_1$ or y_i must be different from $y^*, u, u' \in \mathcal{C}_{M_1}(y_n)$. Choosing $x^* = x_1$, we get B_4 as in Figure 12(a) or we get B_1 (see Figure 12(b)).

If $t = 1$, then either $x^* \notin X_1$ or y_i must be different from $u, u' \in \mathcal{C}_{M_1}(y_1)$. Choosing $y^* = y_n$, we get B_4 as in Figure 12(a), and for the second case, see Figure 12(c). This proves that $[Z_1, Y_1]$ is **II**-free, and, by symmetry, the same is true for each $[Z_i, Y_i]$, $[Z_i, X_i]$ $(i = 1, 2)$.

Let $z_i \in V(G - D)$ $(i = 1, 2)$ be two odd extensions of D adjacent to module M_i of D. We prove that $z_1 z_2 \in E(G)$. Suppose on the contrary that this is not true. For $i = 1, 2$, let $H(n_i)$ be the primitive representative of M_i, and assume that $n = n_1 \geq n_2$. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be the partite sets of $H(n)$ with $x_i y_j \in E(D)$ if and only if $i < j$. Let $x \in X(z_1) \cap \mathcal{C}_{M_1}(x_n)$, $y \in Y(z_1) \cap \mathcal{C}_{M_1}(y_1)$, and $y' \in Y(z_2)$. If M_1 has no edge (that is, $n = 1$), then neither has M_2 (recall that $n \geq n_2$). Then $\mathbf{B} \subset D$ implies that $(D - M_1) - M_2$ contains an edge x^*y^*. Thus we

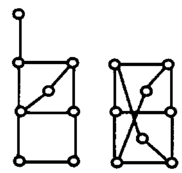

Fig. 14.

get an F_4 (see Figure 13(a)). From now on $n \geq 2$.

Suppose that $D - M_1$ has a vertex nonadjacent with z_2, say, $y^* \notin Y(z_2)$. Let us choose vertices $x^* \in \mathcal{C}_{M_1}(x_1)$ and $x' \in X(z_2)$ adjacent with y^*. (Notice that x' can be chosen from M_2, by Step 2.) Then we get one of F_3 and B_4; see Figure 13(b) if $x^* \notin X(z_1)$ and see Figure 13(c) if $x^* \in X(z_1)$. Thus we may assume that $X(z_2) \cup Y(z_2) = X_2 \cup Y_2$.

Consider the copy of $\mathbf{B}$ defined above, and observe that (in the present case) both x^* and y^* are in M_1. Hence, we may assume that $x^* \in \mathcal{C}_{M_1}(x_1)$ and $y^* \in \mathcal{C}_{M_1}(y_n)$. It follows from $1 < i \leq j < n$ that $x_{j'} \in X(z_1)$ and $y_i \in Y(z_1)$ may be assumed. Since G is $\mathbf{\Delta}$-free, one of x^* and y^* is not adjacent to z, say, $y^* \notin Y(z_1)$. Then, by letting $x = x_j$, we obtain one of F_3 and B_4; see Figure 13(b) if $x^* \notin X(z_1)$ and see Figure 13(c) (with y_i in the role of y_n) if $x^* \in X(z_1)$.

In each case there is a forbidden configuration; therefore, $z_1 z_2 \in E(G)$ follows.

To conclude the proof of (4.2) we refer to the structure theorems in section 2. As we have shown in Steps 1–3, the conditions of (2.7) are satisfied by G; therefore, G has no $\mathbf{\Xi}$, a contradiction. ∎

It is worth noting that our list of forbidden graphs in (4.2) is minimal. Obviously, B_1, B_2, B_3, and B_4 must be on the list; hence each contains $\mathbf{B}$. To see this for F_3 and F_4, in Figure 14 we give connected $\mathbf{\Delta}$-free graphs containing $\mathbf{B}$ such that their only subgraph from the list is F_3 and F_4, respectively.

4.2. Disconnected 4-critical graphs. Let G be a disconnected $\mathbf{\Delta}$-free graph containing $\mathbf{B}$. The connected component $G_0 \subset G$ which contains $\mathbf{B}$ is called the *major component* of G. If G is on-line 3-colorable, then the major component must be $\mathbf{\Xi}$-free by the results in section 4.1. Furthermore, $G - G_0$ is $(K_2 + 2K_1)$- and $\mathbf{II}$-free, since B_7 and B_8 in Figure 4 are not 3-colorable. If $G - G_0$ has no edge, then the algorithm $\mathcal{A}$ in the proof of Theorem 1 is obviously 3-color G.

Therefore, when looking for further 4-critical graphs, one may assume that $G - G_0$ has just one component with an edge, called the *secondary component* of G. Moreover, the secondary component is either a C_5 or a $K_{m,n} - K_2$ or an induced subgraph of $K_{m,n} + K_1$ (i.e., in this last case the secondary component is a complete bipartite graph and G possibly has one more isolated vertex).

(4.3) Let G be a disconnected $\mathbf{\Delta}$-free graph containing $\mathbf{B}$. Then G has on-line chromatic number at most 3 if and only if the major component of G is $\mathbf{\Xi}$-free and G has no induced subgraph isomorphic to any of the graphs B_i, $5 \leq i \leq 10$, in Figure 4.

Proof. All excluded graphs are on-line 4-chromatic (see the beginning of this section); thus we have only to prove that the list is complete. We may assume that G has a major component, a secondary component, and possibly one more isolated vertex. Since B_5 and B_6 are not 3-colorable, we may also assume that the major component G_0 has no $H_1 \subset B_5$ and no $H_2 \subset B_6$ (see Figure 15).

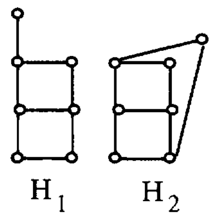

FIG. 15.

First we show that the major component $G_0 \subset G$ is bipartite; moreover, its modules have no edge. To see this, let us consider a maximal bipartite induced subgraph $D \subseteq G_0$ containing $\mathbf{B}$. Clearly, D has at least two nontrivial modules. Let M_1 be a module of D with primitive representative $H(n)$ such that n is as large as possible. Observe that the bipartite complement of H_1 (see Figure 15) is $P_4+K_2+K_1$, and $H(n)$ contains the bipartite complement of P_4+K_1 for $n \geq 3$. Thus we obtain an H_1 in D if $n \geq 3$. Suppose that $n=2$, and notice that $H(2)=\widehat{P_4}$. It is easy to check that $D \supseteq \mathbf{B}$ implies that the bipartite complement of $D-M_1$ contains K_2+K_1. Thus we obtain an H_1 in D also for $n=2$. Therefore, $n=1$ follows; that is, the modules of D have no edge. Since any odd extension of any module of D contains an H_2 (see Figure 15), D obviously has no odd extension. Hence $G_0=D$ follows and concludes the proof of the claim.

Next we define the required on-line 3-colorings depending on the type of the secondary component. Let $z_1, z_2, \dots$ be the order of vertices of G as revealed by Drawer, and let D_k be the colored subgraph after the kth step of the coloring game. For any integer r, let $A(r)$ denote the set of all vertices of G colored with r by an on-line algorithm A.

Case 1. $G-G_0=K_2$. To make the definition on the algorithm easier we introduce two new on-line coloring rules. The *equivalence rule* is as follows: if there are some equivalent vertices with the current vertex z, assign to z the minimum color appearing on a z-equivalent vertex. The *parity first fit rule* (PFF) says that the current vertex should be colored by the smallest color which does not appear on a vertex that is at an odd distance from the current one. We define an algorithm $\mathcal{A}^*$ as follows:

- If z_{k+1} is an isolated vertex, D_k has exactly two components but none of them has three different colors, then color z_{k+1} by 2.
- Otherwise, use the equivalence rule when it applies.
- Otherwise, if the component of z_{k+1} in D_{k+1} is not a complete bipartite graph, then apply the PFF rule.
- In any other cases apply the FF rule with two exceptions:
  - If every neighbor of z_{k+1} is colored by 1, there exist a component in D_k which is 2-colored by 1 and 2, there are no 1- and 3-colored vertices in the same component, and there are no both 1- and 2-colored isolated vertices, then color z_{k+1} by 3.
  - If every neighbor of z_{k+1} is colored by 2, there are no 2- and 3-colored vertices in the same component of D_k and there are no both 1- and 2-colored isolated vertices, then color z_{k+1} by 3.

We have to show that $\mathcal{A}^*$ is a 3-coloring for this class. Let G be a graph with major component G_0 and secondary component K_2 such that the modules of the bipartite graph G_0 have no edges. Let $z_1, z_2, \dots$ be an ordering of the vertices of G,

l is a natural number, and $G_0^l = [X, Y]$ is a subgraph of G_0 induced by $\{z_1, \ldots z_l\}$. It is not too hard to see that when applying $\mathcal{A}^*$ on G with order $z_1, z_2, \ldots$ then after coloring l vertices either X or Y lacks for either any 1-colored or any 2-colored vertices. A similar (but easier) argument shows that the case is similar with 1 and 3 or with 2 and 3. Using these one can check that neither X nor Y can have three different colors. Consequently, there are colors $\{a, b\} \subset \{1, 2, 3\}, a \neq b$, such that X does not have a-colored and Y does not have b-colored vertices. It is obvious that in this case $\mathcal{A}^*$ cannot use more than three colors.

Case 2. $G - G_0 \neq K_2$. In this case $G - G_0$ is a subgraph of either a $K_{m,n} + K_1$ or a $K_{m,n} - K_2$ or a C_5 and either $K_2 + K_1$ or $K_{1,2}$ is contained in it. As B_9 and B_{10} are not contained in G it is easy to see that the edgeless nontrivial modules of G_0 consist of two vertices such that G_0 is a complete bipartite graph with some nonincident edges deleted. We define algorithm $\mathcal{A}^{**}$, which is similar to (but simpler than) $\mathcal{A}^*$, as follows:

- If z_{k+1} is an isolated vertex and D_k has exactly two or exactly three components, then color z_{k+1} by 2.
- Otherwise, use the equivalence rule when it applies.
- Otherwise, if the component of z_{k+1} in D_{k+1} is not a complete bipartite graph, then apply the PFF rule.
- In any other cases apply the FF rule with two exceptions:
  - If every neighbor of z_{k+1} is colored by 1 and there are no 1- and 3-colored vertices in the same component of D_k, then color z_{k+1} by 3.
  - If every neighbor of z_{k+1} is colored by 2 and there are no 2- and 3-colored vertices in the same component of D_k, then color z_{k+1} by 3.

A similar argument as in Case 1 shows that $\mathcal{A}^{**}$ is really a 3-coloring for this class. □

Let us note that by using the algorithms in the proof of (4.3) we obtain that each graph G_j, $j = 5, 6, 7$, in Figure 6 is on-line 3-colorable. Indeed, $\chi_{\mathcal{A}^*}(G_5) \leq 3$, $\chi_{\mathcal{A}^{**}}(G_6) \leq 3$, and $\chi_{\mathcal{A}^{**}}(G_7) \leq 3$ follow.

As a corollary of (4.2) and (4.3) we obtain the list of all 4-critical graphs excluded from on-line 3-chromatic graphs containing **B**: F_3, F_4 in Figure 3, and B_i, $1 \leq i \leq 10$ in Figure 4. This concludes the proof of Theorem 3.

REFERENCES

[G] A. Gyárfás, *On Ramsey Covering Numbers*, Infinite and Finite Sets, Colloq. Math. Soc. János Bolyai 10, North-Holland/American Elsevier, New York, 1975, pp. 801–816.

[GKL1] A. Gyárfás, Z. Király, and J. Lehel, *On-line* 3*-chromatic graphs.* II. *Critical graphs*, Discrete Math., 177 (1997), pp. 99–122.

[GKL2] A. Gyárfás, Z. Király, and J. Lehel, *On-line graph coloring and finite basis problems*, Combinatorics, Paul Erdős is Eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Mathematical Society, Budapest, Hungary, 1993, pp. 207–214.

[GKL3] A. Gyárfás, Z. Király, and J. Lehel, *On-line competitive on-line algorithms*, Technical Report TR-9703-1, Department of Mathematics, Eötvös Loránd University, Budapest, Hungary; also available online from http://www.cs.elte.hu/tr97/.

[GL1] A. Gyárfás and J. Lehel, *On-line and first fit colorings of graphs*, J. Graph Theory, 12 (1988), pp. 217–227.

[GL2] A. Gyárfás and J. Lehel, *First fit and on-line chromatic number of families of graphs*, Ars Combin., 29C (1990), pp. 168–176.

[GL3] A. Gyárfás and J. Lehel, *Effective on-line coloring of P_5–free graphs*, Combinatorica, 11 (1991), pp. 181–184.

[I] S. Irani, *Coloring Inductive Graphs On-line*, in Proceedings of the 31st Annual Symposium on the Foundations of Computer Science, 1990, pp. 470–479.

[K] H. A. Kierstead, *The linearity of first-fit coloring of interval graphs*, SIAM J. Discrete Math., 1 (1988), pp. 526–530.

[K1] H. A. Kierstead, *Recursive and On-Line Graph Coloring*, manuscript,

[K2] H. A. Kierstead, *On-Line Coloring k-Colorable Graphs*, manuscript, 1996.

[KK] H. A. Kierstead and K. Kolossa, *On-line coloring of perfect graphs*, Combinatorica, 16 (1996), pp. 479–491.

[KO] K. Kolossa, *On the on-line chromatic number of the family of on-line* 3*-chromatic graphs*, Discrete Math., 150 (1996), pp. 205–230.

[KPT] H. A. Kierstead, S. G. Penrice, and W. T. Trotter, *On-line and first-fit coloring of graphs that do not induce* P_5, SIAM J. Discrete Math., 8 (1995), pp. 485–498.

[KPT1] H. A. Kierstead, S. G. Penrice, and W. T. Trotter, *On-line coloring and recursive graph theory*, SIAM J. Discrete Math., 7 (1994), pp. 72–89.

[KT1] H. A. Kierstead and W. T. Trotter, *An extremal problem in recursive combinatorics*, Congr. Numer., 33 (1981), pp. 143–153.

[KT] H. A. Kierstead and W. T. Trotter, *On-line graph coloring*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 7, Amer. Math. Soc., Providence, RI, 1992, pp. 85–92.

[L] L. Lovász, *Coverings and colorings of hypergraphs*, Fourth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Utilitas Math., 1973, pp. 3–12.

[LST] L. Lovász, M. Saks, and W. T. Trotter, *An on-line graph coloring algorithm with sublinear performance ratio*, Discrete Math., 75 (1989), pp. 319–325.

[S] D. P. Sumner, *Subtrees of a graph and the chromatic number*, in Theory and Applications of Graphs, G. Chartrand, ed., John Wiley, New York, 1981, pp. 557–576.

[V] S. Vishwanathan, *Randomized online graph coloring*, J. Algorithms, 13 (1992), pp. 657–669.