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Split and balanced colorings of complete graphs

Paul Erdős [a], András Gyárfás [b,*,1]

[a] *Institute of Mathematics, Hungarian Academy of Sciences, Hungary*
[b] *Computer and Automation Institute, Hungarian Academy of Sciences, Hungary*

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Abstract

An (r,n)-split coloring of a complete graph is an edge coloring with r colors under which the vertex set is partitionable into r parts so that for each i, part i does not contain K_n in color i. This generalizes the notion of split graphs which correspond to (2, 2)-split colorings. The smallest N for which the complete graph K_N has a coloring which is not (r,n)-split is denoted by $f_r(n)$. Balanced (r,n)-colorings are defined as edge r-colorings of K_N such that every subset of $\lceil N/r \rceil$ vertices contains a monochromatic K_n in all colors. Then $g_r(n)$ is defined as the smallest N such that K_N has a balanced (r,n)-coloring. The definitions imply that $f_r(n) \leqslant g_r(n)$. The paper gives estimates and exact values of these functions for various choices of parameters. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

A graph whose vertex set can be partitioned into a complete graph and an independent set is called a *split graph*. Split graphs form a self-complementary subfamily of perfect graphs (see for example [3]). A characterization of split graphs in terms of forbidden induced subgraphs has been given by Földes and Hammer ([1], the same result also proved independently in [5]). Split graphs can be also defined in terms of edge colorings of complete graphs as follows. An edge coloring of a complete graph with two colors (red and blue) is a *split coloring* if the vertex set can be covered by the vertices of a monochromatic red and a monochromatic blue subgraph. Clearly, a graph G is a split graph if and only if G and its complement defines a split coloring.

* Correspondence address: MTA, SZTAKI, P.O. Box 63, Kende u. 13-17, 1518 Budapest, Hungary. E-mail: gyarfas@luna.aszi.sztaki.hu.
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This paper generalizes split colorings as follows. An edge coloring of a complete graph K with r colors is called an (r,n)*-split coloring* if the vertices of K can be partitioned into r sets $S_1,\ldots,S_r$ so that S_i has no monochromatic K_n in color i for each i $(1\leqslant i\leqslant r)$. Usual split graphs correspond to $(2,2)$-split colorings. For small complete graphs every coloring is a split coloring, we are interested in the limit where non-split colorings emerge. More precisely, let $f_r(n)$ be the smallest integer m such that there exists an r-coloring of K_m which is not (r,n)-split. Equivalently, $f_r(n)-1$ is the largest m for which every r-coloring of the edges of K_m is (r,n)-split. It is easy to see that $f_2(2)=4$, the 2-coloring of K_4 in which one color class is a four cycle is not a $(2,2)$-split coloring.

Colorings which are not split colorings can be interpreted in Ramsey theory as follows. If a complete graph K is given with an r-coloring which is not (r,n)-split, then for all vertex r-colorings of K we can find a *strongly monochromatic* K_n in K, i.e. K_n whose edges and vertices are all colored with the same color. Clearly, non-split colorings are equivalent with edge colorings where every vertex coloring ensures strongly monochromatic complete subgraphs and $f_r(n)$ is the minimum order of a complete graph for which such a coloring exists. Further connections of non-split colorings and Ramsey numbers are explored in Section 2. Then we shall prove that $f_2(n)=n^2$ (Theorem 2) and $\binom{r}{2}\leqslant f_r(2)\leqslant r^2+r+1$ (Theorems 4, 5).

The non-split property can be enforced by requiring a balanced distribution of colors, defined as follows. An edge r-coloring of K_N is called *balanced* (r,n)*-coloring* if every $A\subseteq V(K_N)$ such that $|A|=\lceil N/r\rceil$ contains a monochromatic K_n in all colors. We define $g_r(n)$ as the minimum N such that K_N has a balanced (r,n)-coloring. Observe that a balanced (r,n) coloring is not an (r,n)-split coloring, therefore

$$f_r(n)\leqslant g_r(n).$$

The above inequality is probably strict for all values of r and n but our estimates are not strong enough to separate them in general (if $r=2$ then Theorems 2, 3 imply this).

It would be interesting to find the asymptotics of $f_r(2)$ and $g_2(n)$. As mentioned before, $f_2(n)=n^2$ and we shall prove that $r^2+1\leqslant g_r(2)\leqslant r^2+r+1$, where the upper bound construction works if a projective plane of order $r+1$ (not r!) exists (Theorem 5). The special cases $r=3$, 4 suggest that the upper bound is the truth ($g_2(2)=5$ seems to be exceptional). This would follow from the following conjecture:

Conjecture 1. If the edges of K_{r^2+1} are colored with r colors then there exist $r+1$ vertices with at least one missing color among them $(r\geqslant 3)$.

The proof of this conjecture for $r=3,4$ is in the last section. From affine planes of order r one can easily construct r-colorings of K_{r^2} in which every set of $r+1$ vertices spans all colors. In fact, $r-1$ color classes can be defined by parallel classes of lines, and one by the union of two parallel classes. This flexibility might suggest that the conjecture is not true.

2. Split critical colorings and Ramsey colorings

We call an edge coloring of a complete graph K (r,n)-split critical if it is not (r,n)-split but it is (r,n)-split on all proper subgraphs of K. The smallest N for which K_N has an (r,n)-split critical coloring is clearly $f_r(n)$. Another interesting split critical coloring is given in Theorem 1. Let $R_r(t)$ denote the classical Ramsey number, i.e. the smallest s for which every r-coloring of the edges of K_s ensures a monochromatic K_t. An (r,t)-Ramsey coloring is an r-coloring of the edges of a complete graph of order $R_r(t)-1$ under which there is no monochromatic K_t.

Theorem 1. *Any $(r,n+1)$-Ramsey coloring is an (r,n)-split critical coloring.*

Proof. Take an $(r,n+1)$-Ramsey coloring C on K_N, where $N=R_r(n+1)-1$. If C is an (r,n)-split coloring on K_N then there is a partition of $V(K_N)$ into A_i, $1 \leqslant i \leqslant r$, such that A_i has no monochromatic K_n in color i. Adding a new vertex y and coloring all edges from y to A_i with color i an edge r-coloring of K_{N+1} is obtained which does not contain monochromatic K_{n+1}. Since $N+1=R_r(n+1)$, this contradicts to the definition of the Ramsey number. On the other hand, for each vertex x of K_N, one can partition $V(K_N)-x$ naturally by the colors of the edges incident to x. Since under C there are no monochromatic K_{n+1}, this partition defines an (r,n)-split coloring. □

Next we give some remarks about (r,n)-split critical colorings. As mentioned before, it is easy to check that $f_2(2,=4)$ and (apart from color switches) there is only one $(2,2)$-split critical coloring on K_4. The $(2,3)$-Ramsey coloring (the pentagon) provides another $(2,2)$-split critical coloring on K_5 (as a special case of Theorem 1). It follows from the split graph characterization theorem [1,5] that there are no other $(2,2)$-split critical colorings.

On 9 vertices there are more than one (probably many) $(2,3)$-split critical colorings, the simplest one is three vertex disjoint triangle in one color, all other edges colored with the other color. (Theorem 2 will show that $f_2(3)=9$ so this example is of minimum order.) Theorem 1 provides a $(2,3)$-split critical coloring on K_{17} from the well known $(2,4)$-Ramsey coloring of K_{17} ([2,4]). It is tempting to conjecture that this is the largest $(2,3)$-split critical coloring (in fact this was stated as a theorem in an earlier version of this paper until a referee pointed to the error in the proof). Now a $(2,3)$-split critical coloring is found on K_{18} and the junior author proved that there are only finitely many $(2,n)$-split critical colorings for every fixed n. (The proof gives huge upper bound for the largest N with a $(2,3)$-split critical coloring on K_N.)

3. On split and balanced $(2,n)$-colorings

Theorem 2. $f_2(n)=n^2$.

Proof. A non-split $(2,n)$-coloring of K_{n^2} is to take n vertex disjoint copies of a red K_n and color all other edges blue. Then a red–blue vertex coloring either colors at least one vertex blue in all copies or colors each vertex of a copy red. In both cases a strongly monochromatic K_n is obtained.

To see that every coloring is $(2,n)$-split on K_m if $m < n^2$, consider an arbitrary red-blue edge coloring of K_m. Select the maximum number of vertex disjoint monochromatic red K_n subgraphs and color with blue each vertex they cover and color all other vertices red. It is immediate that there is no strongly monochromatic K_n in this coloring. □

Theorem 3. $2n(n-1) < g_2(n) \leqslant (2n-1)^2$.

Proof. For the lower bound, we have to show that K_m has no balanced $(2,n)$-coloring if $m \leqslant 2n(n-1)$. Let C be an edge coloring of K_m with red and blue. Select as many vertex disjoint (say t) monochromatic red K_n-s as possible. If their vertices cover at least $m/2$ vertices then select the minimum k such that k of them cover at least $m/2$ vertices. Clearly, $k \leqslant n-1$ therefore the union of the k red K_n do not contain a blue K_n so C is not balanced. On the other hand, if the t red K_n do not cover $m/2$ vertices of K_m then the uncovered part is more than half of the vertices and it does not contain a monochromatic red K_n, so C is not a balanced coloring.

The upper bound follows from considering $2n-1$ vertex disjoint copies of a red K_{2n-1} and coloring all other edges blue. This is a $(2,n)$-balanced coloring of $K_{(2n-1)^2}$. One can clearly improve the upper bound slightly. □

4. On split and balanced $(r,2)$-colorings

Theorem 4. $\binom{r}{2} < f_r(2)$.

Proof. The following claim will be applied repeatedly: if the edges of $K_{\binom{r}{2}}$ are r-colored then some color class defines a graph with at least $r-1$ independent vertices. The proof of this claim for $r \leqslant 4$ is trivial by inspection. For general r, one can show that the minority color class has not enough edges to destroy all independent sets, i.e. it has less edges than the Turán number which is the sum of $\binom{n_i}{2}$ where n_i is the number of elements in the ith class of a balanced partition of $\binom{r}{2}$ into $r-2$ parts. Assuming that $r \geqslant 5$, one can easily determine these numbers n_i for $1 \leqslant i \leqslant r-2$ as follows. For even r, $(r/2)-2$ of the n_i-s are equal to $r/2$ and the other $r/2$ are equal to $(r/2)+1$. For odd r, all but one n_i are equal to $(r+1)/2$ (and one equals to $(r+3)/2$). Thus the claim follows by checking that for $r \geqslant 5$, the number

$$\alpha_r = \frac{\binom{\binom{r}{2}}{2}}{r}$$

satisfies

$$\alpha_r < \left(\frac{r}{2}-2\right)\binom{\frac{r}{2}}{2} + \frac{r}{2}\binom{\frac{r}{2}+1}{2}$$

and also

$$\alpha_r < (r-3)\binom{\frac{r+1}{2}}{2} + \binom{\frac{r+3}{2}}{2}$$

and it is straightforward to check both easily.

The proof of the theorem now follows by induction. Set $N=\binom{r}{2}$. We have to show that an arbitrary r-coloring of the edges of K_N is an $(r,2)$-split coloring. Consider an arbitrary coloring, the claim allows to select a color, say color 1, such that A_1 is an independent set in color 1 and $|A_1|=r-1$. Delete the set A_1 from $V(K_N)$ and a complete graph is left on $\binom{r-1}{2}$ vertices. Change color 1 to color 2 on this complete graph. By induction, this is an $(r-1,2)$-split coloring, i.e. its vertices can be partitioned into sets A_i, $2\leqslant i\leqslant r$ so that A_i has no edge of color i. Adding A_1, we have the required partition for K_N. □

Theorem 5. *$f_r(2)\leqslant g_r(2)\leqslant r^2+r+1$ if a finite projective plane of order $r+1$ exists.*

Proof. Let G_r be the graph which is the union of r vertex disjoint copies of K_r and one copy of K_{r+1} (vertex disjoint from the other components). Our aim is to show that K_{r^2+r+1} can accomodate r edge disjoint copies of G_r. To this end, consider a finite projective plane P of order $r+1$ with two distinguished lines L_1 and L_2 and let x be a point of L_2 not on L_1. Select r points of $L_1\backslash L_2$, say $y_1,\dots,y_r$. Let S be defined by adding x to the set of points in P which are not on the distinguished lines. Note that $|S|=r^2+r+1$. For $i=1,\dots,r$, the graph H_i is defined as follows. The vertex set of H_i is S. The $r+1$ lines in P which are going through y_i and distinct from L_1 partition S into r sets of r elements and one set of $r+1$ elements (the latter is defined by the line of P through y_i and x). Then H_i is the union of complete graphs defined by this partition. Clearly, each H_i is isomorphic to G_r and they are edge disjoint. Coloring the edges of H_i with color i (and coloring uncolored pairs of S arbitrarily) a balanced $(r,2)$-coloring is obtained because all subsets of S which have $\lceil (r^2+r+1)/r\rceil = r+2$ elements, must contain two elements from a block of the $r+1$-partition defined by H_i. □

Next comes the lower bound on $g_r(2)$ which is about twice as good as the lower bound of $f_r(2)$.

Theorem 6. *$r^2+1\leqslant g_r(2)$.*

Proof. The proof is to apply Turán's theorem for the graph spanned by the minority color class. More precisely, let C be an arbitrary r-coloring of the edges of K_{r^2} and assume that the number of red edges is not larger than the number of edges in any other color class. We are going to show that the number of red edges is smaller than $T(r)$, the minimum number of edges needed to destroy all r-element independent sets of a graph on r^2 vertices. Then we conclude that there exist r vertices spanning no red edges, therefore C can not be a balanced $(r,2)$-coloring. By Turán's theorem, $T(r)$ is the number of edges in the graph obtained by partitioning r^2 vertices into $r-1$ complete graphs of nearly equal sizes. This means $r-2$ complete graphs of size $r+1$ and one complete graph of size $r+2$. Thus the following inequality is needed:

$$\frac{\binom{r^2}{2}}{r} < (r-2)\binom{r+1}{2} + \binom{r+2}{2}$$

which is true. This proves that there is no balanced $(r,2)$-coloring of K_{r^2}. One needs a bit more careful calculation to see that there are no balanced $(r,2)$-colorings of K_m for $m < r^2$ but this is omitted. □

5. Small numbers

Proposition 1. $f_3(2) = 8$.

Proof. A 3-coloring of K_8 which is not $(3,2)$-split is defined as follows. Let A be a red cycle with four vertices with two blue diagonals. Take two vertex disjoint copies of A and color all edges joining the copies with green.

It is shown next that any 3-coloring of K_7 is a $(3,2)$-split coloring. Fix an edge coloring with colors 1, 2, 3. If there is a monochromatic K_4, say in color 1 then color its vertices by 2 and it is easy to color the other three vertices by 1 and 3 and avoid strongly monochromatic K_2. Otherwise, if there is a monochromatic triangle T in color 1, select two disjoint edges e_1, e_2 within the other four vertices so that not both of them are colored with 1, assume e_1 is of color 2. Then the vertices of T are colored with 2 and the vertices of e_1, e_2 can be colored with 1, 3 or 3, 1.

Assume that there are no monochromatic triangles. Select a triangle T with color pattern aab. If the set S of the remaining four vertices have two disjoint edges such that one of them has color c or one of them is a and the other is b then it is easy to color the vertices using this partition. Otherwise S can be colored with c and T can be colored with a, b easily. □

Proposition 2. $g_3(2) = 13$.

Proof. Since $10 \leqslant g_3(2) \leqslant 13$ follows from Theorems 5 and 6 it is enough to show that K_{10} has no balanced $(3,2)$-coloring, this is the next lemma. □

Lemma 1. *If K_{10} is colored with three colors then there exist four vertices spanning a K_4 with at least one missing color.*

Proof. The minority color determines a graph G_1 with at most 15 edges so G_1 is either 3-regular or has a vertex x_1 of degree at most 2. In the former case Brooks theorem implies that either G_1 is 3-colorable or contains a K_4, both prove the theorem. Assume the latter case, delete x_1 and its neighbors in G_1, to have a set X such that $|X|\geqslant 7$ and $G_1[X]$ has no three independent vertices. Therefore $G_1[X]$ has a triangle Y. If $Z=X\backslash Y$ has two vertices nonadjacent in G_1 then $G_1[X]$ must contain K_4-e otherwise $G_1[Z]$ is a complete graph. In both cases we have four vertices with a missing color. □

Proposition 3. $g_4(2)=21$.

Proof. Like Proposition 2, but it is more difficult to prove the corresponding lemma (Lemma 2 below). □

Lemma 2. *If K_{17} is colored by four colors then there exist five vertices spanning a K_5 with at least one missing color.*

Proof. Assume that G_1 is the subgraph with the edges of the minority color. This implies that $|E(G_1)|\leqslant 34$ therefore either $\delta(G_1)\leqslant 3$ or G_1 is 4-regular. In the second case Brooks theorem shows that G_1 is either 4-colorable or contains K_5, both possibilities give the required five element vertex set. Thus we may assume that x_1 is a vertex of degree at most 3 in G_1. Let M denote the set of neighbors of x_1 in G_1. An easy count shows that the subgraph of G_1 spanned by $A = V\backslash(x_1\cup M)$ has a vertex x_2 of degree at most 4, let N denote the neighbors of x_2 in $G_1[A]$ and set $X=V\backslash(x_1\cup x_2\cup M\cup N)$. From the construction, $|X|\geqslant 8$. We may assume that $G_1[X]$ has no three independent vertices, otherwise (together with x_1, x_2) we have five independent vertices in G_1 and the theorem is proved. We may also assume the following property (∗): G_1 has no subgraph on five vertices with eight edges.

Case 1: $G_1[X]$ has a $K_4=Y$. Set $Z=X\backslash Y$. We can assume by (∗) that each $z\in Z$ sends at most one edge to Y in color one. This immediately implies that $G_1[Z]$ is complete. This proves the theorem, except when $|X|=8$, and $G_1[Y]$, $G_1[Z]$ are complete graphs. Notice that in this case $|M|=3$, $|N|=4$ must hold. By (∗) we may select u, $v\in N$ nonadjacent in G_1. By (∗) again, both u and v send at most one edge in color 1 to both Y and Z which allows to select two vertices y_1, y_2 in Y and two vertices z_1, z_2 in Z such that none of these four vertices are adjacent in color 1 to u or to v. Using (∗) again, there is a pair y_i, z_j nonadjacent in color 1. This shows that the five vertices x_1, u, v, y_i, z_j are independent in G_1.

Case 2: $G_1[X]$ has no K_4. Since $R(3,4)=9$ and the extremal graph not violating is unique, $G_1[X]$ is an eight cycle with its short chords. As in Case 1, $M=3$, $N=4$. By (∗), select u, $v\in N$ nonadjacent in color 1. The structure of $G_1[X]$ show that at least five vertices of X are needed to block all independent sets of cardinality two in $G_1[X]$. Therefore at least five edges of G_1 run from u, v to X. Our plan is to find at least seven edges of G_1 in $[M,M]$, $[N,N]$, $[M,N]$, $[M,X]$ which together with the

16 edges of $[X, X]$, 5 edges of $[N, X]$, and the 7 edges incident to x_1, x_2 gives 35 edges contradicting the definition of G_1. To achieve this, notice that the degrees of M are at least three and the degrees of N are at least four in G_1 otherwise x_1 or x_2 can be changed to give a better pair. This means that M and $N\setminus\{u, v\}$ have a total of 12 incidences in $V(G_1)\setminus\{x_1, x_2\}$. It is easy to check that the only way to satisfy that with six edges of G_1 is to have a $K_{3,3}$ between M and $N\setminus\{u, v\}$. However, in this case u, v must send six edges to X (instead of the assumed five). This finishes Case 2 and the proof of Lemma 2. □

Proposition 4. $12 \leqslant f_4(2) \leqslant 16$, $13 \leqslant g_2(3) \leqslant 17$.

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