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Vertex colorings with a distance restriction

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Abstract

Let d, k be any two positive integers with $k>d>0$. We consider a k-coloring of a graph G such that the distance between each pair of vertices in the same color-class is at least d. Such graphs are said to be (k,d)-colorable. The object of this paper is to determine the maximum size of $(k,3)$-colorable, $(k,4)$-colorable, and $(k,k-1)$-colorable graphs. Sharp results are obtained for both $(k,3)$-colorable and $(k,k-1)$-colorable graphs, while the results obtained for $(k,4)$-colorable graphs are close to the truth.

Keywords: Coloring; Chromatic number; Distance; Extremal number

1. Introduction

Using the concept of distance, there is a natural way to generalize colorings and the chromatic number. For natural numbers $k,d\geqslant 2$ a k-coloring of the vertices of a graph is called a *d-distant coloring* if $\mathrm{dist}(u,v)\geqslant d$ for each pair of distinct vertices in the same color class. The minimum k for which a graph G has a d-distant k-coloring is denoted $\chi_d(G)$, and is called the *d-distant chromatic number of* G. Clearly, 2-distant colorings are the usual colorings so that $\chi_2(G)=\chi(G)$ for every graph G. For example,

$$\left\lceil \frac{n}{\lfloor n/d \rfloor} \right\rceil$$

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is the d-distant chromatic number of the cycle C_n for $2 \leqslant d \leqslant n$. This d-distant chromatic number of a cycle was considered in an equivalent form as early as 1972 by Kramer [7] and also appears in [10]. Several other articles on d-distant k-colorings have been written by various authors [see 3–6, 8, 9].

It is worth mentioning that the concept of d-distant coloring can be expressed by using one (of the many) definitions of graph powers. If G^t is the graph obtained from G by joining all pairs of distinct vertices which are at distance at most t then it is immediate that G is d-distant k-colorable if and only if G^{d-1} is k-colorable (in the usual sense).

There are some trivial d-distant k-colorable graphs we wish to avoid. If $n \leqslant k$ then any graph on n vertices is d-distant k-colorable for every d and this remains true for graphs whose components have at most k vertices. To avoid these trivial colorings *we define (k,d)-colorable graphs as those connected graphs which have more than k vertices and have d-distant k-colorings*. We shall also use (k,d)-colorings for d-distant k-colorings.

The purpose of this paper is to address the following extremal problem: what is the maximum number of edges in (k,d)-colorable graphs with n vertices? This maximum is denoted by $f(n,k,d)$ and the graphs attaining the maximum are called extremal graphs. Usually we keep k and d fixed and let n tend to infinity.

We note first that the transition from $d=2$ to $d \geqslant 3$ changes the character of d-distant colorings. If $d \geqslant 3$ then the set of edges between any two color classes of a (k,d)-coloring must be pairwise disjoint. This observation relates d-distant colorings to acyclic colorings, introduced by Grünbaum [2], where the set of edges between any two color classes must form an acyclic subgraph. Clearly, the acyclic chromatic number of a graph is not larger than its d-distant chromatic number (for any $d \geqslant 3$). The order of magnitude of n in $f(n,k,d)$ also changes with the transition. Observe that the maximum number of edges in a $(k,2)$-colorable graph is precisely the Turán number. The unique extremal graph is the Turán graph (the complete k-partite graph with evenly distributed vertex set), so (assuming that k divides n) $f(n,k,2) = n^2(k-1)/2k$. However, as the next theorem shows, $f(n,k,d)$ is linear in n if $d \geqslant 3$.

Theorem 1. *Assume that $d \geqslant 3$. Then the maximum degree of a (k,d)-colorable graph is at most $k-d+2$. Consequently,*

$$f(n,k,d) \leqslant \frac{(k-d+2)n}{2}.$$

Proof. Assume that G is (k,d)-colorable and consider a (k,d)-coloring of G. Let v be a vertex of G and let A denote its neighbors in G. Clearly, $\{v\} \cup A$ is colored with distinct colors because $d \geqslant 3$. This shows that $|A| \leqslant k-1$ and since G is connected and $|V(G)| > k$, there is a connected subgraph H of G with $k+1$ vertices such that $\{v\} \cup A \subset V(H)$. From the Pigeonhole Principle, there exist two vertices $x, y \in V(H)$ in the same color class. Let P be a shortest path connecting x, y in H. Since P can intersect the star $\{v\} \cup A$ in at most three vertices, the length of P is at most $2+(k+1)-(|A|+1)=k+2-d_G(v)$. Since the length of P is at least d, it follows that $d \leqslant k+2-d_G(v)$, i.e. the degree of v is at most $k-d+2$, proving the theorem. □

Theorem 1 and its proof have the following immediate consequences.

Corollary 1. *If G is (k,d)-colorable then $k \geqslant d$.*

Corollary 2. *If $d \geqslant 3$ then the (d,d)-colorable graphs are the paths of length at least d and the cycles of length sd for $s \geqslant 2$.*

Based on the cases covered so far, it will be assumed throughout (unless otherwise stated) that $n>k>d>2$. Next we define a natural candidate for the extremal graphs. For our purposes it is enough to consider those values of n for which k divides n. The *necklace* $N(n,k,d)$ for $n=ks$ is defined as the following graph. Take s vertex disjoint copies $G_1,\dots,G_s$ of the complete graph K_{k-d+3}. Remove an edge v_iw_i from each G_i. Then connect w_i to v_{i+1} for $1 \leqslant i \leqslant s$ (with $v_{s+1}=v_1$) by a path P_i of length $d-2$ so that the set of inner vertices of the P_i's are disjoint from each other and also from the vertices of the G_i's. It is easy to check that $N(n,k,d)$ has $n=ks$ vertices, $(n/k)(\binom{k-d+3}{2}+d-3)$ edges, and is a (k,d)-colorable graph. Notice that for $k=d$ the necklace is a cycle, in accordance with Corollary 2.

For $d=4$, there is another useful graph which can be obtained by modifying the necklace and then adding some attachments to it. To describe the attachments consider st vertex disjoint copies $H_{i,j}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$ of the complete graph K_{k-1}, where $t \leqslant k-4$. For each i and j select $t+2$ special vertices $a_{i,j}^1, a_{i,j}^2, \dots, a_{i,j}^{t+1}$, $u_{i,j}$ of $H_{i,j}$ and remove edges $u_{i,j}a_{i,j}^{\ell}$ for $\ell=1,2,\dots,t+1$. Let $z_1, z_2, \dots, z_s$ be s additional vertices disjoint from the vertex set of each $H_{i,j}$. Every $H_{i,j}$ is joined to z_i by the edge $u_{i,j}z_i$ to form a graph L_i for each i $(1 \leqslant i \leqslant s)$. Each of the graphs L_i will be appropriately attached to a modified necklace described below.

To modify the original necklace we need to remove additional edges from each of the graphs $G_1, G_2, \dots, G_s$ used in its construction. For a fixed t, $t \leqslant k-4$, select t vertices $b_i^1, b_i^2, \dots, b_i^t$ in G_i, different from both v_i and w_i, and remove in addition to edge v_iw_i edges $v_ib_i^{\ell}$, $w_ib_i^{\ell}$ for $\ell=1,2,\dots,t$. For each path P_i of length 2 used in the construction let y_i denote its middle vertex. The constraint $t \leqslant k-4$ guarantees the modified necklace is connected. With these modifications to the necklace, form the graph $NA(n,k,4,t)$ by identifying vertices y_i and z_i for all i, keeping the graphs L_i and the modified necklace otherwise vertex disjoint. We refer to this graph $NA(n,k,4,t)$ as a *necklace with attachments*. Observe

that $NA(n,k,4,0)=N(n,k,4)$, a usual necklace.

One can easily check that $NA(n,k,4,t)$ is $(k,4)$-colorable, has

$$s\left[(t+1)\binom{k-1}{2}-t(t+1)-(2t+1)+(t+2)\right]$$

edges, and

$$s[(t+1)(k-1)+1]$$

vertices. Therefore

$$\max_{0\leqslant t\leqslant k-4}\frac{|E(NA(n,k,4,t))|}{|V(NA(n,k,4,t))|}=\max_{0\leqslant t\leqslant k-4}\frac{(t+1)\binom{k-1}{2}-t^2-2t+1}{(t+1)(k-1)+1}$$

which produces a better lower bound than the necklace for $\sup_n f(n,k,4)/n$ (at least for large k). However, this example cannot be generalized for general d. Indeed, it is likely

that for $d\geqslant 5$ the lower bound of $\sup_n f(n,k,4)/n$ given by the necklace cannot be significantly improved.

The next theorem gives $f(n,k,3)$ for every n and shows that the necklace is an extremal graph for $d=3$ (if k divides n).

Theorem 2. *Assume $n=qk+r$ where $0\leqslant r\leqslant k-1$. Then*

$$f(n,k,3)=\binom{k}{2}q+\binom{r}{2}.$$

Proof. Let G be a $(k,3)$-colorable graph with the color classes $X_1, X_2,\ldots,X_k$ and $|X_1|\leqslant|X_2|\leqslant\cdots\leqslant|X_k|$. For each pair of X_i and X_j $(1\leqslant i<j\leqslant k)$ the edges between X_i and X_j form an independent edge set since every pair of vertices of X_i (X_j) are at distance $\geqslant 3$. Thus, there are at most $|X_i|$ edges between X_i and X_j, which implies the following.

$$e(G)\leqslant(k-1)|X_1|+(k-2)|X_2|+\cdots+2|X_{k-2}|+|X_{k-1}|\leqslant\binom{k}{2}q+\binom{r}{2}.$$

Equality holds throughout the above expressions when $|X_i|=q$ for $i=1,2,\ldots,k-r$, $|X_i|=q+1$ for $i=k-r+1,\ldots,k$, and there is an appropriate matching from X_i to X_j for all $i<j$ making G connected. □

The next result is for the case $d=k-1$ and it is sharp if k divides n. Its proof is given in Section 2. (The uncovered case, $k=5$, is covered in Theorem 4.)

Theorem 3. *If $k\geqslant 6$ then $f(n,k,k-1)\leqslant n(k+2)/k$. The necklace $N(n,k,k-1)$ shows that equality is possible.*

Our most difficult result concerns the case $d=4$.

Theorem 4. *For $k\geqslant 4$*

$$\frac{\binom{k-1}{2}+1}{k}\leqslant\sup_n\left(\frac{f(n,k,4)}{n}\right)\leqslant\max_{0\leqslant t\leqslant k-3}\frac{\binom{k-1}{2}+1+t(\binom{k-1}{2}-t)}{(t+1)(k-1)+1}.$$

The proof of Theorem 4 will be given in Section 3, here we only give some comments. The lower bound is obtained from the necklace $N(n,k,4)$. The complicated expression in the upper bound comes from the proof attempt that the necklace is frequently extremal. Notice that the term in the upper bound with $t=0$ is precisely the lower bound, so the theorem is sharp and the necklace is really extremal when the maximum is attained at $t=0$. This happens for $k=4$ (in accordance with Corollary 2) and also for $k=5,6$ (when the upper bound is $\max\{\frac{7}{5},\frac{4}{3},\frac{15}{13}\}=\frac{7}{5}$ and $\max\{\frac{11}{6},\frac{20}{11},\frac{27}{16},\frac{32}{21}\}=\frac{11}{6}$). This gives the following corollary.

Corollary 3. $\sup_n(f(n,5,4)/n)=\frac{7}{5}$ *and* $\sup_n(f(n,6,4)/n)=\frac{11}{6}$. *The necklaces* $N(n,5,4)$ *and* $N(n,6,4)$ *give equality.*

The bounds in Theorem 4 are separated if $k\geqslant 7$. It was thought at first that the gap is due to the proof method and the lower bound is the truth, i.e. the necklace is always an extremal graph. However the proof of the upper bound lead to the construction of the necklace with attachments which improves the lower bound (at least for large k).

We conclude that it is probably very difficult to find $f(n,k,4)$ for $k\geqslant 7$ even for the case when k divides n. Notice though that the gap between the upper and lower bound in Theorem 4 is less than $\frac{1}{2}$. In fact, the lower (upper) bound can be approximated by $(k-3)/2\,((k-2)/2)$ for large values of k.

2. Proof of Theorem 3

We will prove Theorem 3 by contradiction. Suppose G is a $(k,k-1)$-colorable graph with n vertices and more than $(k+2)n/k$ edges. Further, we assume that G is a counterexample of Theorem 3 with the minimum number of vertices. Since G is a $(k,k-1)$-colorable graph, the maximum degree of G is at most 3 by Theorem 1. Since $e(G)>(k+2)n/k$, G must contain a vertex of degree three with all of its neighbors of degrees at least 2. Since $k\geqslant 6$ and $|V(G)|\geqslant k+1$, easy counting shows that G contains a vertex induced connected subgraph H with $k+1$ vertices such that H has a vertex z of degree 3 and all its neighbors have degree at least 2 in H.

Let $V_1,V_2,\ldots,V_k$ be a partition of the vertices of G which gives G a $(k,k-1)$-coloring. We will simply refer to each V_i as the color class determined by color i. By the Pigeonhole Principle, H contains two vertices x and y which are in the same color-class, say in V_1. Let $P[x,y]$ be one of the shortest paths in H joining x and y. Since $\mathrm{dist}_G(x,y)\geqslant k-1$, $P[x,y]$ contains at least k vertices. Because $d_H(z)=3$, $P[x,y]$ contains exactly k vertices. Let w be the vertex of H which is not on the path $P[x,y]$.

Note that w must be either the vertex z or one of the neighbors of z. If $w=z$, then since $P[x,y]$ is of minimal length w is adjacent to three consecutive vertices on the path $P[x,y]$. By the minimality of the length of $P[x,y]$, we have that if $w\neq z$ then w is either adjacent to three consecutive vertices on $P[x,y]$, or two consecutive vertices on $P[x,y]$, or two vertices of distance two on the path $P[x,y]$. In particular, the following result holds.

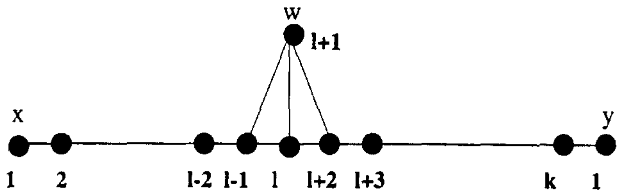

Fig. 1.

Claim 1. *H is a subgraph of the graph shown in Fig. 1 with the possibility that one edge from the vertex w to $P[x,y]$ is removed. Each vertex $v_i \in V_i$ for $1 \leqslant i \leqslant k$ and $y = v_{k+1} \in V_1$.*

Note that in Fig. 1, the possible neighbors of w are $v_{\ell-1}$, v_ℓ, and $v_{\ell+2}$. We also assume that we choose H such that, $\min\{\ell, k-\ell\}$ as large as possible, that is, trying to place v_ℓ in the 'approximate' middle of the path $P[x,y]$. In the following, we will show that we can assume that x and y are the only two vertices in H having possible neighbors outside H. To do so, let u be a vertex of $V(H) - \{x,y\}$ having a neighbor v outside H. Without loss of generality, we assume that

$$\mathrm{dist}_H(u,x) \leqslant \mathrm{dist}_H(u,y).$$

Suppose that $v \in V_i$. Since $\mathrm{dist}_G(v,v_i) \geqslant k-1$, the only possibility is that $u = w = v_{\ell+1}$ and $v \in V_k$ and $N(w) \cap V(P[x,y]) = \{v_1, v_2\}$. Since $k \geqslant 6$ and G is a $(k,k-1)$-colorable graph, in this case, we either have that x does not have neighbors outside H or both $u = v_2$ and $x = v_1$ have a common neighbor $v \in V_k$ outside H. In both cases, we let H^* be the graph induced by $V(H) \cup \{v\} - \{y\}$. Then the only two vertices which may have neighbors outside H^* are v and v_k. Thus, without loss of generality, we can assume that the only vertices in H which may have neighbors outside H are x and y.

Clearly, H does not contain $(k+2)(k+1)/k$ edges. Hence, $n > k+1$. In the following, we will study the structure of H and vertices nearby. Then, we will show that $e(G) \leqslant (k+2)n/k$ if $n \leqslant 2k$ or that there is a counterexample to Theorem 3 with smaller order (which is impossible by assumption).

2.1. *Suppose that $k+1 < n \leqslant 2k$*

In this case, we will show that G does not contain both edges $v_{\ell+1}v_{\ell-1}$ and $v_{\ell+1}v_{\ell+2}$. Suppose, to the contrary, $v_{\ell+1}v_{\ell-1} \in E(G)$ and $v_{\ell+1}v_{\ell+2} \in E(G)$. If both x and y have neighbors outside H, we can extend H step by step to show that G contains the graph shown in Fig. 2 as a vertex induced subgraph.

Note, if we cannot extend H to the above graph, then at some step the terminal vertex must have degree 1, which contradicts our assumption that G is a counterexample with the minimum number of vertices. Since $n \leqslant 2k$, it is readily seen that G is a subgraph of the graph in Fig. 3, which implies that $e(G) \leqslant (k+2)n/k$, a contradiction.

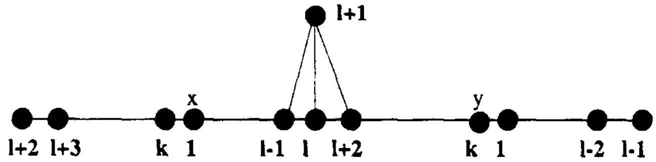

Fig. 2.

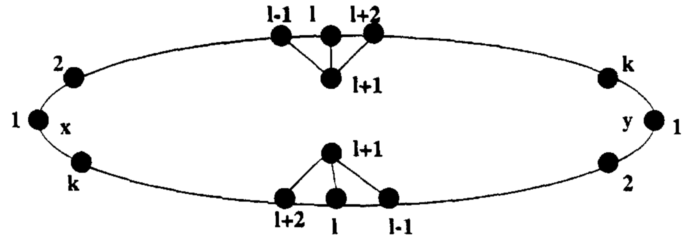

Fig. 3.

Hence, one of x and y does not have a neighbor outside H. Without loss of generality, we assume that x is the one. Since G is a counterexample with the minimum number of vertices, the minimum degree $\delta(G) \geqslant 2$. Thus, $x = v_{\ell-1}$, that is, $\ell = 2$. Note that x does not have a neighbor outside H implies that y must have a neighbor outside H. Since G is a $(k, k-1)$-colorable graph, the neighbors of y outside H can only be in $V_2 \cup V_3$. If y has only one neighbor y^* outside H, say in V_2, let G^* be a graph obtained from the graph G by removing the vertex x and adding an edge v_3v_5. It is readily seen that G^* is a $(k, k-1)$-colorable graph and $|V(G^*)| = n - 1 \geqslant k + 1$ and

$$e(G^*) \geqslant e(G) - 1 > \frac{(k+2)(n-1)}{k},$$

which contradicts the minimality of the number of the vertices of G. Therefore, y has two neighbors y^* and y^{**} outside H. Since G is a $(k, k-1)$-colorable graph and $k \geqslant 6$, the neighbors of y^* and y^{**} outside $V(H) \cup \{y^*, y^{**}\}$ must be in V_4 and the neighbors must be the same one if both of them have a neighbor outside H. Using the property that any pair of vertices in the same color class must have distance at least $k-1$ and G has at most $2k$ vertices, we can show that G is a subgraph of the graph shown in Fig. 4.

Then, it is readily seen that $e(G) \leqslant (k+2)n/k$, a contradiction.

Hence, one of edges $v_{\ell+1}v_{\ell-1}$ and $v_{\ell+1}v_{\ell+2}$ is missing in H. Without loss of generality, we assume that the edge $v_{\ell+1}v_{\ell-1}$ is missing, that is, $d(v_{\ell+1}) = 2$ and $v_{\ell+1}v_{\ell}$ and $v_{\ell+1}v_{\ell+2}$ are edges of G. In the same manner as above, we can show that if both x and y have neighbors outside H, then G contains the vertex induced subgraph shown in Fig. 5, where the dotted lines indicate possible edges, and the two end vertices in $V_{\ell+1}$ at the two ends may be the same.

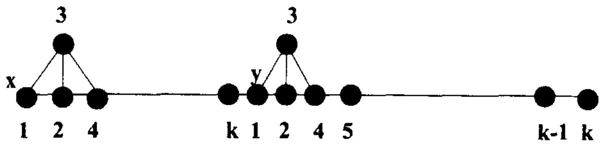

Fig. 4.

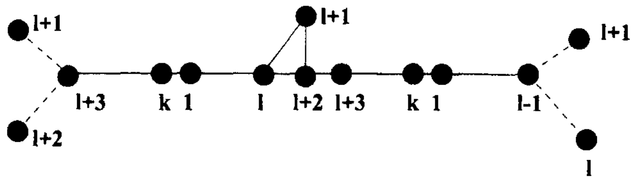

Fig. 5.

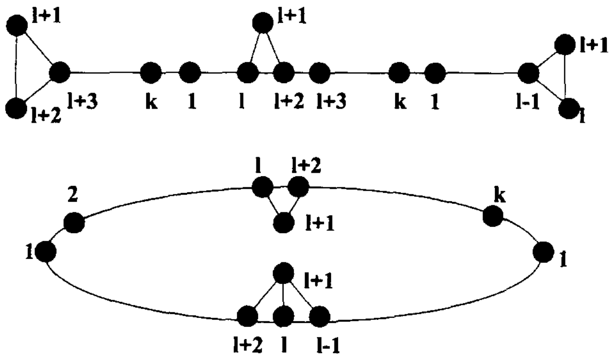

Fig. 6.

Since G has no more than $2k$ vertices, it is readily seen that G is a subgraph of one of the two graphs shown in Fig. 6.

In either case, we have $e(G) \leqslant (k+2)n/k$, a contradiction.

Thus, we can assume that x does not have a neighbor outside of H. Then, y must have a neighbor outside of H. Let G^* be the graph obtained from G by removing the vertex x and adding an edge v_2v_4. Since v_k does not have a neighbor outside of H, we see that G^* is also a $(k,k-1)$-colorable graph with $e(G^*) > (k+2)(n-1)/k$, which contradicts the minimality of the number of vertices of G.

2.2. Suppose that $n \geqslant 2k+1$

Claim 2. *Both x and y have neighbors outside H.*

Proof. Suppose, to the contrary, x does not have a neighbor outside H. Then, y is the only vertex of H which may have neighbor outside H. In particular, we see that

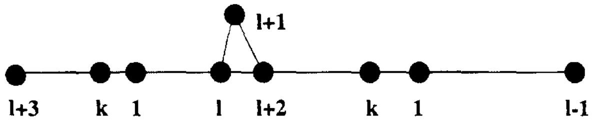

Fig. 7.

$G-(V(H)-\{y\})$ is connected. Note that $H-y$ is incident to at most $k+2$ edges. Thus,

$$e(G) \leqslant e(G-(V(H)-y))+(k+2) \leqslant \frac{(k+2)n}{k},$$

a contradiction. □

If x has two neighbors x_1^* and x_2^* outside H, then we can assume that $x_1^* \in V_k$ and $x_2^* \in V_{k-1}$ and we have either $\ell = k-1$ or $\ell = k$ since $V_1, V_2, \ldots, V_k$ give a $(k, k-1)$-coloring of G. Since $k \geqslant 6$, y has exactly one neighbor y^* outside H which lies in V_2. In this case, the subgraph induced by $V(H) \cup \{y^*\} - \{x\}$ will contradict the maximality of $\max\{\ell, k-\ell\}$. Thus, x has exactly one neighbor $x^* \in V_k$ outside H and y has exactly one neighbor $y^* \in V_2$ outside H. We will consider the subgraph $G^* = (G - V(H-y)) \cup \{x^* y\}$, that is, by removing all vertices in H except y and adding an edge $x^* y$.

Clearly, G^* is connected. If G^* is a $(k, k-1)$-colorable graph, then

$$e(G) \leqslant e(G^*) - 1 + (k+3) \leqslant \frac{(k+2)n}{k},$$

a contradiction.

Therefore, there are two vertices u and v in $V(G^*)$ and they are in the same color class, say V_i, such that $\mathrm{dist}_{G^*}(u, v) \leqslant k-2$. Since $V_1, V_2, \ldots, V_k$ give a $(k, k-1)$-coloring of G, u and v must be in different components of $G - V(H-y)$. Without loss of generality, we assume that u and x^* are in the same component and v and y are in the same component in $G - V(H-y)$. If $i \neq \ell+1$ or $i = \ell+1$ and H contains both edges $v_{\ell+1}v_{\ell-1}$ and $v_{\ell+1}v_{\ell+2}$, then $\mathrm{dist}_G(x, v_i) + \mathrm{dist}_G(y, v_i) = k-1$. Since $\mathrm{dist}_G(u, v_i) \geqslant k-1$ and $\mathrm{dist}_G(v, v_i) \geqslant k-1$, then

$$\mathrm{dist}_G(u, x) + \mathrm{dist}_G(y, v) \geqslant 2(k-1) - (k-1) = k-1.$$

Thus,

$$\mathrm{dist}_{G^*}(u, v) = \mathrm{dist}_G(u, x^*) + 1 + \mathrm{dist}_G(y, v) \geqslant k-1,$$

a contradiction. Therefore $i = \ell+1$ and one of the edges $v_{\ell+1}v_{\ell-1}$ and $v_{\ell+1}v_{\ell+2}$ is missing. Without loss of generality, we assume that $v_{\ell+1}v_{\ell-1}$ is missing. Then, G contains the graph shown in Fig. 7 as a vertex induced subgraph.

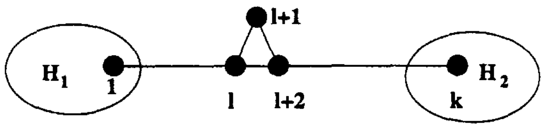

Fig. 8.

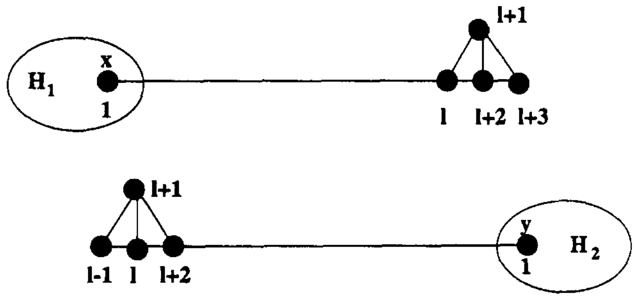

Fig. 9.

Since $G - V(H - y)$ is a disconnected graph, the graph G has the structure shown in Fig. 8, where H_1 and H_2 are connected subgraphs. Now we form two new connected graphs G_1 and G_2 as shown in Fig. 9.

Clearly, $|V(G_1)| \geqslant k + 1$ and $|V(G_2)| \geqslant k + 1$. It is also not difficult to check that both G_1 and G_2 are $(k, k-1)$-colorable graphs. Then,

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) - 7 \\ &\leqslant \frac{k+2}{k}|V(G_1)| + \frac{k+2}{k}|V(G_2)| - 7 \\ &= \frac{k+2}{k}(n+5) - 7 \\ &\leqslant \frac{k+2}{k}n, \end{aligned}$$

since $k \geqslant 6$, a contradiction, completing the proof of Theorem 2. □

3. Proof of Theorem 4

Define $f(k,4) = \sup_n f(n,k,4)/n$. The lower bound is provided by the necklace $N(n,k,4)$. To prove

$$f(k,4) \leqslant \max_{0 \leqslant t \leqslant k-3} \frac{\binom{k-1}{2} + 1 + t(\binom{k-1}{2} - t)}{(t+1)(k-1)+1},$$

let

$$g(k) = \max_{0 \leqslant t \leqslant k-3} \frac{\binom{k-1}{2} + 1 + t(\binom{k-1}{2} - t)}{(t+1)(k-1)+1}.$$

Clearly, $g(k) \geqslant (\binom{k-1}{2} + 1)/k$, the value at $t = 0$.

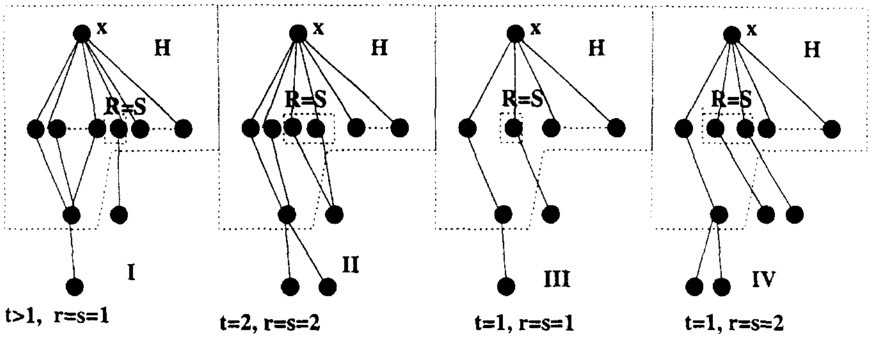

Fig. 10.

Suppose, to the contrary, G is a $(k,4)$-colorable graph with $n \geqslant k+1$ vertices and more than $g(k)n$ edges. Further, we assume that G has the minimum number of vertices with the given properties. Suppose the vertex partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ gives a $(k,4)$-coloring of G, that is, if u and v are in the same V_i for some i then $\text{dist}(u,v) \geqslant 4$. Let H be a subgraph of G. As usual, we will let $e(H)$ denote the number of edges of G with both ends in H and $\partial(H)$ denote the number of edges with one end in H and the other one is not in H. We will use $\partial^*(H)$ for $e(H)+\partial(H)$, that is, the total number of edges incident with H. We will prove the theorem by highlighting the following claims. First, we notice that the maximum degree $\Delta(G) \leqslant k-2$ by Theorem 1.

Claim 3. *G contains a connected induced subgraph H of k vertices such that*
1. *Some vertex x of H has degree $k-2$,*
2. $\partial^*(H) \leqslant \binom{k-1}{2} + 2$, *and*
3. *if equality holds in 2, then H is one of the graphs shown in Fig. 10 (including the connections into the the rest of G as shown).*

Proof. Since $e(G) > (\binom{k-1}{2}+1)n/k$, G contains a vertex of degree $k-2$. Let x be one such vertex. Without loss of generality, we assume that $x \in V_1$ and the neighbors of x are in $V_2, \ldots, V_{k-1}$ respectively. Since G is connected and G has at least $k+1$ vertices, there exist vertices in V_k which have neighbors in $N(x)$. Let y be one of those vertices such that $|N(y) \cap N(x)|$ is maximum.

Note that $N[x] \cup \{y\}$ contains a vertex in each color class. Assume

$$N[x] \cup \{y\} = \{x = x_1, x_2, \ldots, x_{k-1}, x_k = y\},$$

where $x_i \in V_i$ for each $i = 1, 2, \ldots, k$.

Let $T = N(x) \cap N(y)$ and

$$S = \{x_i \mid N(y) \cap V_i \neq \emptyset \text{ and } N(x) \cap N(y) \cap V_i = \emptyset\},$$

that is, S is the set of the neighbors of x for which y has a neighbor in the same color-class outside $N[x]$.

Let R be the set of vertices in $N(x) - T$ having neighbors outside $N[x]$. Since G is a $(k,4)$-colorable graph, all neighbors of R outside H are in V_k.

Since every pair of vertices in the same color-class are at a distance at least 4, there is no edge from $S \cup R$ to T. There is no edge inside R if $|T| = 1$ by the maximality of $|T|$. Let $s = |S|$, $t = |T|$, and $r = |R|$. Now $N[x]$ contains $k-1$ vertices and a total of $\binom{k-1}{2}$ possible edges. Thus, the number of edges incident to H is at most

$$\binom{k-1}{2} + s + t + r - t \times |S \cup R|.$$

Since G is connected and $|V(G)| \geqslant k+1$, then $|S \cup R| \geqslant 1$.

If $t \geqslant 3$, then $s+t+r-t \times |S \cup R| < 2$ unless $s = r = 1$ and $S = R$. Therefore, Claim 3 is true in case $t \geqslant 3$.

If $t = 2$, then $t + s + r - t \times |S \cup R| = 2 + s + r - t \times |S \cup R| \leqslant 2$, with equality holding only if $S = R$ and R induces a complete subgraph, and the later implies that $|N(R) - N[x]| = 1$. In particular, $r \leqslant 2$ by the maximality of t. Therefore, Claim 3 is true in case $t = 2$.

If $t = 1$, the number of edges incident to H is at most

$$\binom{k-1}{2} + s + t + r - t \times |S \cup R| - \binom{r}{2}$$

$$\leqslant \binom{k-1}{2} + 2 - |R - S| + \left(r - \binom{r}{2} - 1\right).$$

Note that $-|R-S| + (r - \binom{r}{2} - 1) \leqslant 0$ and the equality holds if and only if either $r = 1$ or $r = 2$ and $R \subseteq S$. Further, by the maximality of $|T|$, each vertex in R is adjacent to different vertices if $|R| = 2$. Therefore the claim follows. □

Our purpose is to investigate the structure of $G - V(H)$ which will lead to a contradiction to the minimality of the number of vertices of G. Basically, we will divide the remaining proof into two main cases $\partial^*(H) < \binom{k-1}{2} + 2$ or $\partial^*(H) = \binom{k-1}{2} + 2$. In the first case, we will simply remove the subgraph H from G. In the later case, we need to remove the vertices of H and add one or two edges to the remaining graph and the following claim is needed.

Claim 4. *G contains a vertex induced subgraph H with k vertices with the maximum degree $\Delta(H) = k-2$ and one of the following three conditions holds:*

1. $\partial^*(H) < \binom{k-1}{2} + 2$.
2. $\partial^*(H) = \binom{k-1}{2} + 2$ *and there exists an edge $e \notin E(G)$ such that the graph $(G - V(H)) \cup e$ has the property that each pair of vertices in the same color-class are at distance at least* 4.

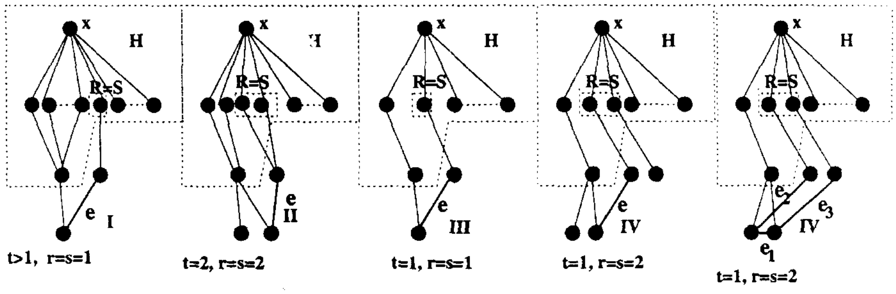

Fig. 11.

3. $\partial^*(H) = \binom{k-1}{2} + 2$ *and there exist three edges* e_1, e_2 *and* e_3 *such that* $e_1 \in E(G)$ *and* $e_2, e_3 \notin E(G)$ *and* $(G - V(H) - e_1) \cup \{e_2, e_3\}$ *has the property that each pair of vertices in the same color-class are at distance at least* 4.

Proof. Let H be a vertex induced subgraph of G guaranteed by the previous claim. Claim 4 clearly follows if $\partial^*(H) < \binom{k-1}{2} + 2$. Assume that $\partial^*(H) = \binom{k-1}{2} + 2$. Then, H is one of the four graphs shown by the previous claim. In Fig. 11, we indicate where the edge e will be added to $G - V(H)$.

It is readily seen that in cases I–III, $(G - V(H)) \cup \{e\}$ has the property that every pair of vertices in the same color-class are at distance at least 4. In the following we will show that if there is no subgraph such that condition (2) holds in case IV of Claim 3, then condition (3) of Claim 4 holds. Without loss of generality, we assume the graph H has been labeled as shown in Fig. 12, where the numbers indicate the color-classes to which the vertices belong.

Since $\partial^*(H) = \binom{k-1}{2} + 2$, y has exactly two neighbors outside of H. Since G is a $(k,4)$-colorable graph with the color-classes $V_1, V_2, \ldots, V_k$ and $G(N[x])$ contains every possible edge from x_2 to x_i except x_3 and x_4, where we assume that two neighbors of y outside H are in V_3 and V_4 respectively. Let $H^* = G(N[x] \cup \{y^*\})$ and $H^{**} = G(N[x] \cup \{y^{**}\})$ where y^* and y^{**} are the neighbor x_3 and x_4 respectively outside H. If $\partial^*(H^*) < \binom{k-1}{2} + 2$, the claim follows. Thus, we assume $\partial^*(H^*) = \binom{k-1}{2} + 2$. In particular, we have that y^* has exactly two neighbors outside H^* and they are in V_2 and V_3 respectively. In the same manner, we can show that y^{**} has exactly two neighbors outside of H^{**} and they are in V_2 and V_3, respectively. Let

- $z_3 \in V_3$ and $z_4 \in V_4$ be two neighbors of y outside of H;
- $z_2^* \in V_2$ and $z_4^* \in V_4$ be two neighbors of y^* outside of H^*;
- $z_2^{**} \in V_2$ and $z_3^{**} \in V_3$ be two neighbors of y^{**} outside of H^{**}.

Since H does not satisfy condition (2), $G_1 = (G - V(H)) \cup \{z_3 y^*\}$ contains two vertices u and v in the same color-class at distance at most 3. Clearly, one of u and v must be in $\{z_3, z_4, z_2^*, z_4^*\}$. Since $V_1, V_2, \ldots, V_k$ give a $(k,4)$-coloring of G, u and v must be in V_4. Then, $\{u, v\} = \{z_4, z_4^*\}$. In particular, we have $z_3 z_4 \in E(G)$. Similarly, we can show that $z_2^* z_4^* \in E(G)$ and $z_2^{**} z_3^{**} \in E(G)$. Let $e_1 = z_3 z_4$, $e_2 = z_3 y^*$, and $e_3 = z_4 y^{**}$. Then, G contains graph shown in Fig. 13 as a subgraph.

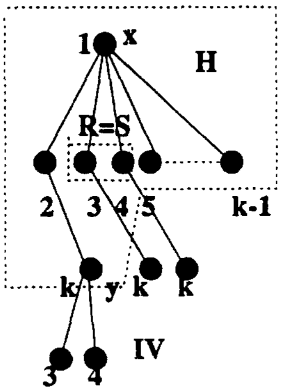

Fig. 12.

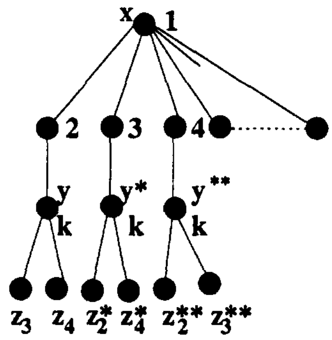

Fig. 13.

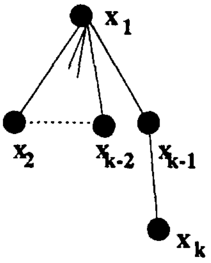

Fig. 14.

It is readily seen that $(G - V(H) - e_1) \cup \{e_2, e_3\}$ satisfies condition (3). □

Let H be the vertex induced subgraph guaranteed by Claim 3. We let $G \ominus H$ denote

- $G - V(H)$ if $\partial^*(H) < \binom{k+2}{2} + 2$,
- $(G - V(H)) \cup \{e\}$ if $\partial^*(H) = \binom{k+2}{2} + 2$ and condition 2 is met,
- $(G - V(H) - e_1) \cup \{e_2, e_3\}$ if $\partial^*(H) < \binom{k+2}{2} + 2$ and condition 3 is met.

The components of $G \ominus H$ are called *good components* if either the component is of order $>k$ or if the edge density of these components $\leqslant g(k)$. Components with larger density and order $\leqslant k$ are called *bad components*. Note that since we have chosen G to be a smallest order graph (of order $\geqslant k+1$) where the density exceeds $g(k)$, all components of order $>k$ have density $\leqslant g(k)$. Furthermore, without loss of generality, we also assume that $G \ominus H$ has a minimum number of bad components. Since G is a counterexample, there must be at least one bad component in $G \ominus H$.

Claim 5. *Let $L_1 = K_{1,k-2}$ and $L_2 = K_{1,k-2}$ be two vertex disjoint stars in G and let L_3 be the graph shown in Fig.* 14 *and vertex disjoint from L_2.*

Then,

1. *at most one end vertex of L_1 is adjacent to an end vertex of L_2, and*
2. *if x_k is adjacent to some set of end vertices of L_2, then no x_i $(1 \leqslant i \leqslant k-1)$ is adjacent to an end vertex of L_2.*

Proof. (i) Label the vertices of

$$V(L_1) = \{y_1, y_2, \ldots, y_{k-1}\}$$

where y_1 is the center vertex of L_1 and let

$$V(L_2) = \{z_1, z_2, \ldots, z_{k-1}\}$$

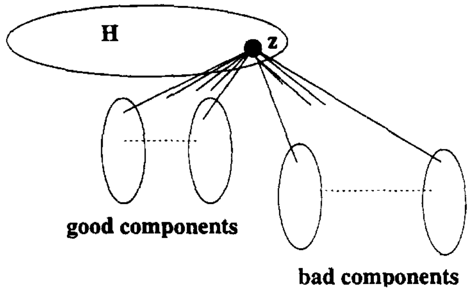

Fig. 15.

where z_1 is the center vertex of L_2. Suppose there are at least two edges $y_i z_j$ and $y_\ell z_t$ with $1 \leqslant i, j, \ell,\ t \leqslant k-1$, and $i \neq \ell$ or $j \neq t$. Assume $j \neq t$. Then z_j and z_t each receives a color different from the k colors assigned to the vertices of L_1. But there are at most $k-1$ different colors, so that z_j and z_t receive the same color, which contradicts G is a $(k,4)$-colorable graph with the coloring $V_1, V_2, \ldots, V_k$.

(ii) Note that x_k has a color distinct from those assigned to vertices of L_2 since x_k is adjacent to some set of end vertices of L_2. But then if some x_i, $2 \leqslant i \leqslant k-1$, were adjacent to a vertex z_t of L_2, then x_i also receives a color different from these of L_2 implying that x_i and x_k have the same color. This contradicts $\mathrm{dist}_G(x_i, x_k) \geqslant 4$. □

Claim 6. *Each bad component of $G \ominus H$ has $k-1$ vertices and G has a cutvertex z such that G has the structure shown in Fig.* 15.

Proof. By definition a component is bad if it has edge density $> g(k) > (\binom{k-1}{2}+1)/k$. Clearly, any component of order $\leqslant k-2$ has edge density $\leqslant (k-3)/2 < (\binom{k-1}{2}+1)/k$, so that such components are good. Also, from the proof of Claim 3, we see that no component of order k can have more than $\binom{k-1}{2}+1$ edges and be connected to a vertex of H, which implies that all components of order k are good.

Let B_1 be a bad component in $G \ominus H$ and let H contain L_3 as shown in Fig. 14. Since B_1 has edge density $> (\binom{k-1}{2}+1)/k$, B_1 contains a vertex y of degree $k-2$. Then, $N[y]$ spans B_1. Let L_2 be a spanning star of B_1 with center at y. Let $V(B_1) = \{y = y_1, y_2, \ldots, y_{k-1}\}$. Applying Claim 4, $L_2 \cup L_3$ spans one of the graphs shown in Fig. 16.

We claim in either case z is a cutvertex of G and all the other vertices in $N(x_1)$ are not cutvertices. If not, the graph spanned by $\{z, y_1, y_2, \ldots, y_k\}$ can replace H and the deletion of this graph with fewer bad components than $G \ominus H$, a contradiction. □

Let G^* be the graph obtained by deleting from G all good components indicated in the above claim. Then G^* is one of the graphs shown in Fig. 17.

Note that there can only be one bad component in the second possibility shown above since all neighbors of z outside H have the same color.

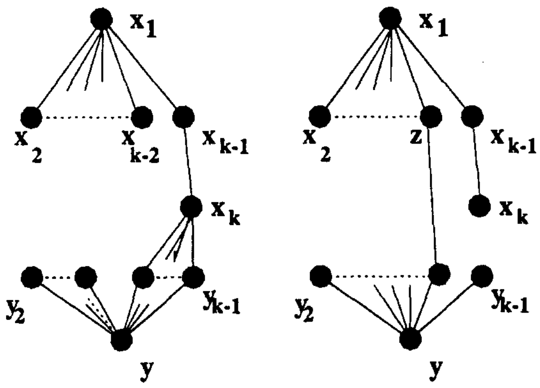

Fig. 16.

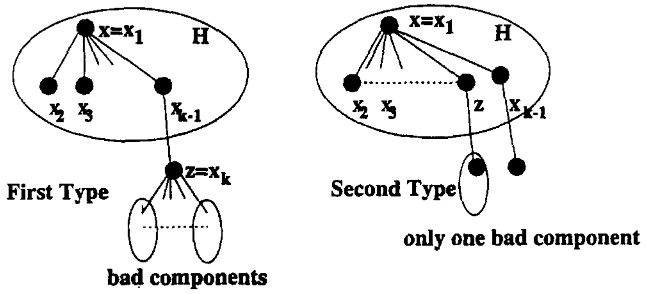

Fig. 17.

Claim 7. *The inequality* $e(G^*) \leqslant g(k)n$ *holds.*

Proof. We first show when G^* (of the second type) shown above has a maximum number of edges, then it is isomorphic to one of the first type. Split $\{x_2,\dots,x_{k-1}\}-\{z\}$ into three sets A, D, C as shown in Fig. 18 where C is the set of the neighbors of x_k and hence nonneighbors of z, A is the set of remaining nonneighbors of z, and D is the set of the rest vertices in $N(x)$. Let B denote the one bad component connected to z. Further, let $|C|=j$, $|A|=q$, and $|D|=m$.

Observe that the $k-2$ vertices of $B-\{b\}$ must receive their colors from the distinct set of $k-1$ colors given to $A\cup D\cup C\cup\{x_1,x_k\}$. Also, each of the vertices of $B-\{b\}$ which are assigned the colors of $D\cup\{x\}$ must be nonadjacent to b. Hence G^* (of the second type) has at most

$$\binom{k-1}{2}-j-q+j+\binom{k-1}{2}-m$$

edges and this is maximized by choosing $m=q=0$ which makes that G^* be of the first type. We only need to maximize the number of edges for G^* in this case.

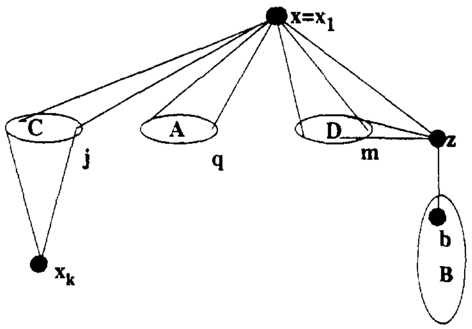

Fig. 18.

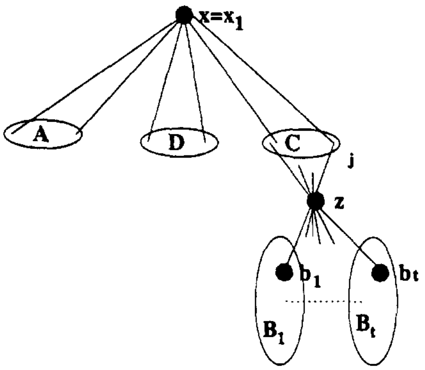

Fig. 19.

Split $N(x)$ into three sets A, D, and C as shown in Fig. 19 where C is the set of the neighbors of z and no vertex in C is adjacent to a vertex in A, while some vertex of C is adjacent to each vertex of D. Let $B_1, B_2, \ldots, B_t$ denote the bad components. Let $j = |C|$.

Observe that the colors given to $\{b_1, b_2, \ldots, b_t\}$ (see Fig. 19) must be a subset of the colors assigned to vertices of A so that $|A| \geqslant t$. Also any neighbor of b_i in B_i must have a color different from each color given to b_j, $j \neq i$, and also different from the colors of $C \cup \{z\}$. Thus, since each $B_i - \{b_i\}$ has $k-2$ vertices, there must be at least $j+t-1$ vertices in $B_i - \{b_i\}$ which are not adjacent to b_i. (Here we have also used that all colors of B_i are distinct and the same as those k colors given to $\{x\} \cup A \cup D \cup C$.) Therefore, G^* has maximum number of edges when the only nonadjacencies are those indicated in the above discussion. Hence the maximum number of edges is G^* is (choose $|A| = t$)

$$e(G^*) = \binom{k-1}{2} - tj + j + t + t\left(\binom{k-1}{2} - (t+j-1)\right)$$

for appropriately chosen t and j.

Clearly, $e(G^*)$ will reach its maximum when $j=1$ (Assuming $t>0$). Thus, assuming $t>0$, the density $e(G^*)/((t+1)(k-1)+1)$ is maximized by maximizing

$$\frac{\binom{k-1}{2}+1+t(\binom{k-1}{2}-t)}{(t+1)(k-1)+1}$$

where $0<t\leqslant k-3$. □

Since the G^* is obtained from G by deleting the good components of $G\ominus H$, the density of G is no more than the density of G^*, which contradicts the assumption that $e(G)>g(k)n$. Therefore Theorem 4 follows.

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