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Discrete Applied Mathematics 85 (1998) 139–147

DISCRETE APPLIED MATHEMATICS

A communication problem and directed triple systems

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Received 5 September 1995; received in revised form 15 July 1997; accepted 8 December 1997

Abstract

The solution of a specific network problem is shown to be equivalent to the decomposition of a certain complete symmetric digraph into edge disjoint balanced transitive triples. Further related decomposition results and conjectures are presented as well. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Networks; Transitive triples; Complete symmetric digraph; Pairable graph

1. Introduction

Let $G_{A,B}$ be a balanced bipartite graph where each vertex represents a processor and each edge a link (or communication line) between processors in sets A and B, $|A| = |B| = m$. For each of the $2m$ processors select a different t edge star, identifying the center of the star with the processor, keeping these graphs otherwise vertex disjoint. Thinking of the $2tm$ end vertices of the resulting graph as terminals one wishes to pair the end vertices attached to the vertices of A with those attached to vertices of B in an arbitrary manner and ask the following question. Under what conditions do there exist edge disjoint paths between all tm pairs of vertices, i.e. when can all paired terminals communicate simultaneously? Clearly, the edge density of the graph $G_{A,B}$ must be known as well as the magnitude of t and m.

Alternately, one can describe the above pairing by a graph, called a *demand graph*, formed as follows: if t^* of the terminals adjacent to a_i are paired with t^* of the terminals adjacent to b_j, then the demand graph which represents this pairing is a bipartite multigraph with t^* the multiplicity of edge a_ib_j. Thus each demand graph describes a specific pairing of the terminals. Note that each such demand graph is a multigraph which is t-regular.

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[1] Research supported by OTKA Grant 7309.
[2] Research partially supported by NSF Grant DMS-9400530.

PII S0166-218X(98)00028-6

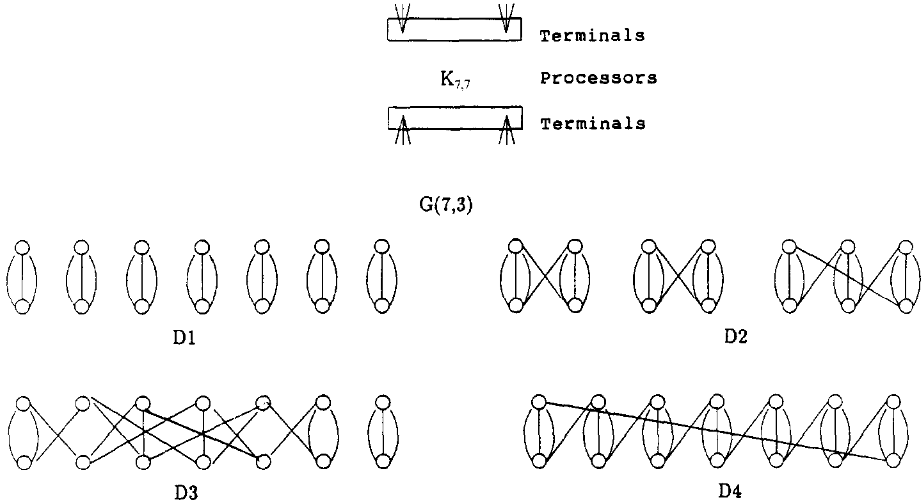

Fig. 1. $G(7,3)$ and some demand graphs.

Since the general network problem described above is very difficult, the first objective of this article is to focus on the case when $G_{A,B}=K_{m,m}$ and the underlying demand graph has a certain regular configuration. It will be seen that a solution in this special case occurs precisely when one can decompose a related complete symmetric digraph into a collection of edge disjoint transitive triples, where a transitive triple centered at vertex a is a digraph with vertex set $\{a,b,c\}$ and edge set $\{ba,ac,bc\}$.

The specifics of these ideas require further discussion. Let $G(m,t)$ denote the graph obtained after the addition of the stars to $G_{A,B}=K_{m,m}$. Call the graph $G(m,t)$ *pairable* whenever the edge disjoint paths exist for all possible t-regular demand graphs. Pairable graphs have also been considered in [2–4].

There is an obvious necessary condition for $G(m,t)$ to be pairable; it is that $m \geqslant 3(t-1)+1$. To see this suppose the demand graph (with parts A and B) has edge set a_ib_i $(i=1,2,\ldots,m)$, each edge of multiplicity t. This demand graph requires that there be t edge disjoint paths between a_i and b_i for all i, and that these t paths collectively use at least $3(t-1)+1$ edges of $K_{m,m}$.

For example, when $m=7$ and $t=3$, under the pairing just described, each a_i in $K_{7,7}$ is joined to b_i by 3 edge disjoint paths, one path a single edge and the other two each with at least 3 edges. This requires that collectively the paths use at least $7(1+2\cdot 3)=7\cdot 7$ edges, i.e. all edges of $K_{7,7}$. The graph $G(7,3)$ together with the demand graph D_1 used in the pairing just discussed, as well as several other demand graphs are shown in Fig. 1.

Similarly when $t=2p+1$ $(2p+2)$ then $m\geqslant 6p+1$ $(6p+4)$ so that all edges of $K_{6p+1,6p+1}$ $(K_{6p+4,6p+4})$ are used if the required path condition is met. It is seen (Theorem 1 given in Section 2) that the appropriate collection of paths exist under the pairing described. As mentioned above the existence is given in terms of a factorization of the complete symmetric digraph into transitive triples with center distribution determined by the underlying demand graph.

In light of the proof of Theorem 1 in terms of a factorization of a digraph, Section 2 is devoted to two other similar decomposition problems. Theorem 2 is a generalization of Theorem 1 to complete digraphs with multiple directed edges and Theorem 3 gives a similar decomposition for noncomplete digraphs.

The last section (Section 3) uses a greedy algorithm to prove a weaker form of Conjecture 1. Also another related question is introduced there. In summary, the objectives of this article are to

(1) relate a special case of the path pairable problem to a factorization of the complete symmetric digraph into equally distributed transitive triples,
(2) explore further other similar factorizations, and
(3) to present several beautiful related open questions and conjectures.

In light of the above discussion a principal conjecture of the paper is the following one.

Conjecture 1. *Both* $G(6p+1,2p+1)$ *and* $G(6p+4,2p+2)$ *are pairable graphs.*

2. Restricted demand graphs and balanced triples

There is a natural one-to-one correspondence between the family of balanced multigraphs with parts A and B, $|A|=|B|=m$, and multidigraphs on m vertices. This correspondence is used to identify special 3-edge paths in $K_{m,m}$ with directed transitive triples in $\vec{K}_m$ (the complete symmetric digraph on m vertices).

To be more specific let $A=\{a_1,a_2,\ldots,a_m\}$ and $B=\{b_1,b_2,\ldots,b_m\}$ denote the vertex classes of a balanced multibipartite graph. Let $G_{A,B}$ be such a bipartite graph and let G_C be a digraph with vertex set $C=\{c_1,c_2,\ldots,c_m\}$ and edge set defined as follows: c_ic_j is a directed edge of G_C if and only if a_ib_j is an edge of $G_{A,B}$. Clearly this determines a natural correspondence between the family of balanced bipartite multigraphs with parts of size m and the family of multidigraphs on m vertices.

Of particular interest is this correspondence when $G_{A,B}=K^-_{m,m}$, where $K^-_{m,m}$ is the graph obtained by deleting the 1-factor $a_1b_1,a_2b_2,\ldots,a_mb_m$ from $K_{m,m}$. In this case the corresponding digraph is $\vec{K}_m$. Further it is of special interest that under this correspondence the 3-edge path $a_ib_ja_kb_i$ in $K^-_{m,m}$ is associated with the transitive triple c_kc_i,c_ic_j,c_kc_j centered at c_i. Thus also in the graph $G(m,t)$ all 3-edge paths from a_i to b_i are, under the above correspondence, associated with transitive triples centered at c_i in $\vec{K}_m$.

One should recall the special pairing for $G(6p+1,2p+1)$ ($G(6p+4,2p+2)$) mentioned in the Introduction. The pairing has demand graph (with parts A and B) and edge set a_ib_i ($i=1,2,\ldots,m$), each edge of multiplicity t. But under this pairing $G(6p+1,2p+1)$ ($G(6p+4,2p+2)$) pairable means $K^-_{6p+1,6p+1}$ ($K^-_{6p+4,6p+4}$) is factorable into 3-edge paths with $2p\ (2p+1)$ of the paths joining a_i to b_i for each i. By the above-mentioned correspondence this is equivalent to the factorization of $\vec{K}_m$ into transitive triples with $2p\ (2p+1)$ of the triples centered at each vertex. One calls such a factorization *balanced*. The existence of such factorizations are known [1] where the construction was based on cyclic Steiner triple systems. A short but different proof is included here.

The proof uses the following result (which generalizes Hall's theorem and is a special case of the f-factor theorem on bipartite graphs [6]): If $G=(A,B)$ is a bipartite graph in which for every $S \subseteq A$, $|\Gamma(S)| \geqslant t|s|$ (t is fixed) then there exists a 'perfect t-star matching' from A to B, i.e. G contains a subgraph which is the union of vertex disjoint t-stars, each centered at vertices of A.

Theorem 1. *The digraph $\vec{K}_m$ has a balanced factorization into transitive triples when $m=6p+1$ or $6p+4$.*

Proof.

Case 1: $m=6p+1$.

Consider the Steiner triple system (*STS*) $\mathscr{H}$ on $\{1,2,\ldots,6p+1\}$ points. It is claimed that $\mathscr{H}$ can be factored in $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_{6p+1}$, where each $\mathscr{H}_i$ consists of p triples intersecting at i. To see this consider the bipartite graph expressing the point versus triple incidences. This bipartite graph has a factor consisting of the vertex disjoint union of $6p+1$ stars each with p edges. This follows since each set of t points is incident to at least pt triples (generalization of Hall's theorem, [6, p. 50 Problem 16]). But each triple of $\mathscr{H}_i$ can be replaced by $\vec{K}_3$. Since $\vec{K}_3$ can be factored into two transitive triples centered at the same vertex, each $\mathscr{H}_i$ can be replaced by $2p$ transitive triples centered at i.

Case 2: $m=6p+4$.

Let $\{0,1,2,\ldots,6p+3\}$ be the vertex set of $\vec{K}_{6p+4}$. Consider a resolvable *STS* $\mathscr{H}$ on $\{1,2,\ldots,6p+3\}$ and let $\mathscr{H}_0$ be a paralled class of $\mathscr{H}$. For each triple $H \in \mathscr{H}_0$ add $\{0\}$ to the points of H and replace these four points by a balanced factorization of $\vec{K}_4$ into transitive triples. (The reader can easily check that such a balanced factorization of $\vec{K}_4$ exists.) By the same argument used when $m=6p+1$, the triples in $\mathscr{H}-\mathscr{H}_0$ can be factored into $\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_{6p+3}$ where each $\mathscr{H}_i$ consists of p triples intersecting at i. But each triple in $\mathscr{H}_i$ can be replaced by two transitive triples centered at i, completing the proof. □

One should comment that the construction for the above proof given in [1] gives a regular factorization into a directed triple system (DTS), where regular means that each vertex appears the same number of times at each of the three possible positions on the set of all transitive triples. It is easy to see that a balanced factorization of $\vec{K}_m$ into transitive triples is equivalent to a regular factorization into a DTS. For non-complete digraphs the concepts are not equivalent, with a balanced factorization less restrictive.

It is natural to consider the problem analogous to the decomposition of Theorem 1 when each directed edge of $\vec{K}_m$ is replaced by multiple directed edges. It is easy to check that if such decompositions are to hold for values of m different from $6p+1$ and $6p+4$, then the multiplicity t must be divisible by 3. Thus, it suffices to consider balanced transitive triple systems on $\vec{K}_m^3$, the complete graph of order m with each edge replaced by three oriented edges in both directions. The next theorem establishes that such decompositions are possible.

Theorem 2. *The graph $\vec{K}_m^3$ has a balanced factorization into transitive triples for $m \geqslant 3$.*

Proof. For $m=6p+1$ or $m=6p+3$ the solution is obvious: one takes a STS on m points and replace each triple $\vec{K}_m^3$ by a balanced factorization into transitive triples (BFTT). Also if $m=6p+4$ then take the BFTT guaranteed by Theorem 1 and repeat it two additional times.

$m=6p$: A set of m elements is partitioned into $A \cup B$ with $|A|=3$ and $|B|=6p-3$. Set $A=\{a_1,a_2,a_3\}$. Next, take a resolvable STS $\mathscr{H}$ on B and let $\mathscr{H}_i$ for $i=1,2,3$, be three parallel classes of $\mathscr{H}$. This requires $p \geqslant 2$, the case $p=1$ $(m=6)$ is listed as an exceptional case later. For each $i \in \{1,2,3\}$ and for each triple $H \in \mathscr{H}_i$ add i to the three points of H and replace this four element set by a BFTT of $\vec{K}_4^3$. Finally, replace A and also all triples of $\mathscr{H} - \bigcup_{i=1}^{3} \mathscr{H}_i$ by a BFTT of $\vec{K}_3^3$.

$m=6p+2$: The previous argument is applied with $|A|=5$, $|B|=6p-3$, $A=\{a_1,a_2,\dots,a_5\}$. Now, $\mathscr{H}_i$ (for $i=1,2,\dots,5$) are parallel classes of a resolvable STS on B. This requires $p \geqslant 3$, the classes $p=1,2$ $(m=8,14)$ are exceptional and handled later. Also a BFTT of $\vec{K}_5^3$ is needed to replace A, so $m=5$ is also exceptional.

$m=6p+5$: The previous argument is again applied with $|A|=8$, $|B|=6p-3$. Here at least eight parallel classes are needed in the resolvable STS on B which requires $p \geqslant 4$ and adds $m=11,17,23$ to the list of exceptions. (The BFTT of $\vec{K}_8^3$, needed to replace A, will be available and is already exceptional since $m=6p+2$ was the previous case.)

Exceptional Cases: $(m=5,6,8,11,14,17,23)$.

The only exceptional cases not considered below are when $m=6$ or 11. These were omitted since they were proved by special constructions which do not relate nicely to the remainder of the proof.

$m=5$: Each four element subset of a five element set is replaced by a BFTT of $\vec{K}_4$.

$m=8$: Set $A=\{a\}$, $|B|=7$, $A \cap B=\emptyset$. Let $\mathscr{H}$ be a BFTT on B (using Theorem 1) and repeat this BFTT of $\mathscr{H}$ a second time. Finally for each $H \in \mathscr{H}$, the three points of H and $\underline{a}$ are replaced by $\vec{K}_4$ with a BFTT.

The cases $m=14,17,23$ use the following result: if $6k+1$ is a prime power then there is a STS $\mathscr{H}$ on $6k+1$ points which is factorable into $\mathscr{H}_1,\mathscr{H}_2,\dots,\mathscr{H}_k$ so that each point is in exactly three triples of $\mathscr{H}_i$, $i=1,2,\dots,k$. This is called a *good partition of $\mathscr{H}$*. In fact, these good partitions are only used for $k=2$ and $k=3$ (See [5, Theorem 15.3.4], actually #9 and #29 are needed from Appendix I of [6]).

$m=14$: Set $A=\{a\}$, $|B|=13$, $A\cap B=\emptyset$. Let $\mathscr{H}$ be a STS on B with a good partition $\mathscr{H}=\mathscr{H}_1\cup\mathscr{H}_2$. For each $H\in\mathscr{H}_1$ the three points of H and $\underline{a}$ are replaced by $\vec{K}_4$ with a BFTT. Let α be a one-to-one map between B and the triples of $\mathscr{H}_2$, such that each point is incident to its image (from the regularity of $\mathscr{H}_2$). Replace each triple of $\mathscr{H}_2$ with $\vec{K}_3$ centered at $\alpha^{-1}(H)$ and repeat this system two additional times. Apply the same method to $\mathscr{H}_i$ and repeat it a second time.

$m=17$: Set $A=\{a,b,c,d\}$, $|B|=13$, $A\cap B=\emptyset$. Let $\mathscr{H}$ be a STS on B with a good partition $\mathscr{H}=\mathscr{H}_1\cup\mathscr{H}_2$. For each $H\in\mathscr{H}_1$, $H\cup a$ and $H\cup b$ are replaced by $\vec{K}_4$ with a BFTT. Similarly, for each $H\in\mathscr{H}_2$, $H\cup c$ and $H\cup d$ are replaced by $\vec{K}_4$ with a BFTT. The set A is replaced by $\vec{K}_4^3$ with a BFTT. This last part of the decomposition is similar to the case $m=14$: each triple $H\in\mathscr{H}_1$ $(H\in\mathscr{H}_2)$ is replaced by $\vec{K}_3$ so the common centers are placed at different points of B.

$m=23$: This is similar to the previous case. Let $A=\{a,b,c,d\}$, $|B|=19$, $A\cap B=\emptyset$. Select a STS $\mathscr{H}$ on B with a good partition $\mathscr{H}_1\cup\mathscr{H}_2\cup\mathscr{H}_3$. For each $H\in\mathscr{H}_1$, $H\cup a$ and $H\cup b$ are replaced by $\vec{K}_4$ with a BFTT. Similarly for each $H\in\mathscr{H}_2$, $H\cup c$ and $H\cup d$ are replaced by $\vec{K}_4$ with a BFTT. The set A is replaced by $\vec{K}_4^3$ with BFTT. The last part of the decomposition is the same as the case $m=17$, the only difference is that $\mathscr{H}_3$ is repeated two additional times. □

The pairing which lead to the decomposition of $\vec{K}_m$ in Theorem 1 was determined by a demand graph consisting of a factor of multiedges, each with a multiplicity of about $m/3$. Thus suppose one changes the demand graph on m vertices to one which remains regular of degree $m/3$ and also still has all its parallel edges on a factor with the multiplicity of these edges lowered. In light of Theorem 1 this suggests the following conjecture.

Conjecture 2. *Let D be a digraph on m vertices with minimum in–out degree $\geqslant 2m/3+t-1$ $(t\leqslant m/3)$. Then D contains a subdigraph consisting of edge disjoint transitive triples with $t-1$ of them centered at each of its m vertices.*

At this point the conjecture can only be proved when $t\leqslant\sqrt{m}/12$ by a greedy algorithmic approach.

Theorem 3. *Let D be a digraph of order m (m sufficiently large) with minimum in–out degree $\geqslant 2m/3+t-1$ and with $t\leqslant\sqrt{m/12}$. Then D contains a subdigraph of edge disjoint transitive triples with $t-1$ centered at each of the m vertices.*

The proof of the theorem depends upon the following lemma.

Lemma. *Let D be a digraph of order m with minimum in–out degree $\geqslant 2m/3+(t-1)-2(t-1)i$ (t and i fixed integers). Select an arbitrary subset $X\subseteq V(D)$ such that $|X|\leqslant(1/(2t-1))\,(m/3-4(t-1)i+3)$. Then there exist $|X|$ vertex disjoint sets, each set consisting of $t-1$ edge disjoint transitive triples centered at a different vertex of X and otherwise vertex disjoint.*

Proof. Assume one has constructed k of the vertex disjoint sets of $t-1$ triples which uses exactly k vertices of X, each centered at a different vertex of X and otherwise vertex disjoint. Let W denote the set of vertices of these k sets of $t-1$ triples. If $k<|X|$ select an unused vertex $x \in X-W$ from X and let $y_1, y_2, \ldots, y_{t-1} \in V(D)-(W \cup X)$ such that $y_i x \in E(D)$ for all i $(i=1,2,\ldots,t-1)$. Set $Y=\{y_1, y_2, \ldots, y_{t-1}\}$ and let $U=\{u \in V(D)-(W \cup Y \cup X): xu \in E(D)\}$ Finally set $Z=V(D)-(W \cup Y \cup U \cup X)$.

One should observe that since the elements of Z are not in the outset of x, $|Z| \leqslant m-1-\delta^+(x) \leqslant m/3+2(t-1)i-t$. Therefore, the family of k vertex disjoint sets of $t-1$ triples can be enlarged to a $(k+1)$st set of triples as long as there are $t-1$ different out neighbors for the $t-1$ distinct elements of Y in the set U. But each vertex of Y is possibly outadjacent to other members of Y, members of X not in W, and to all of $Z \cup W$. Hence, the family can be enlarged if $(t-2)+(|X|-k)+(2t-1)\,k+m/3+2(t-1)i-t+t-2<2m/3-2(t-1)i+t-1$. This is equivalent to

$$k+(1/(2t-2))|X|<(1/(2t-2))(m/3-4(t-1)i+3). \tag{1}$$

But $|X| \leqslant (1/(2t-1))(m/3-4(t-1)i+3)$ implying $|X|+(1/(2t-2))|X|=((2t-1)/(2t-2))\,|X| \leqslant (1/(2t-2))(m/3-4(t-1)i+3)$. Therefore if $k<|X|$ inequality (1) holds so that the set W can be enlarged at long as $|X| \leqslant (1/(2t-1))(m/3-4(t-1)i+3)$. □

Proof of Theorem 3. The theorem will follow by repeated application of the lemma, starting with $i=0$ (as long as $|X| \neq \emptyset$). Assume one repeats the lemma for $i=0,1,\ldots,r$. Then the theorem will follow if r and t are such that

$$(1/(2t-1))\sum_{i=0}^{r}(m/3-4(t-i)i+3) \geqslant m \tag{2}$$

with the only restriction (from the proof of the lemma) that both $2m/3+(t-1)-2(t-1)i>0$ and $m/3-4(t-1)i+3>0$ for $i=0,1,\ldots,r$. Therefore, $r \leqslant (1/(4t-4))(m/3-3)$ and the reader can check that (2) holds for $t \leqslant \sqrt{m}/12$. □

There is an additional variation of both Theorems 1 and 3 which should be true. Suppose that the edges of a 1-regular subgraph are removed from $\vec{K}_{6p+4}$, in other words, both the indegree and outdegree of the resulting digraph D is $(6p+2)$. In this case does D contain $(6p+4)(2p)$ edge disjoint transitive triples such that $2p$ are centered at each vertex. Note that this is one less triple per vertex than found for $\vec{K}_{6p+4}$ and is the best possible. Although such a decomposition is likely it appears to be difficult. It is easy to prove in the special case when $6p+4$ is of the form $12l+4$ and when the deleted 1-regular subgraph is such that each component is a pair of oppositely directed edges on the same pair of vertices.

Proposition. *Let D be the digraph described above with $|D|=12l+4$. Then D has $(12l+4)(4l)$ edge disjoint transitive triples with $4l$ centered at each vertex.*

Proof. Since $v = 12l + 4$, there exists a resolvable design with $k = 4$ and $\lambda = 1$ on $12l + 4$ points. This gives a decomposition of $12l + 4$ into 4-sets with each pair of objects in exactly one 4-set such that the 4-sets can be decomposed into $4l + 1$ parallel classes, each class consisting of $3l + 1$ disjoint 4-sets. But each 4-set can be replaced by a $\vec{K}_4$ which is itself decomposable into four transitive triples one centered at each vertex. Eliminating the directed edges resulting from one of the parallel classes leaves a digraph contained in D which has the desired decomposition. □

3. General demand graphs

In all the results presented in the last section the demand graphs considered when proving $G(m,t)$ pairable were of a restricted nature. Nevertheless, it seems as though the pairing considered in the results of Theorem 1 suggests the appropriate general bound on t in terms of m. Thus, it was conjectured (Conjecture 1) that $G(m,t)$ is pairable for all $t \leqslant m/3$. This appears to be difficult, but it is established for smaller values of t.

Theorem 4. *The graph $G(m,t)$ is pairable for $t \leqslant m/12$.*

Proof. Consider any pairing of the mt end vertices incident to vertices of A with those mt incident to vertices of B, and let D be the demand graph associated with this pairing. Further let $m(u,v)$ denote the multiplicity of an unordered pair uv in D, $u \in A$, $v \in B$, and define $M(D) = \sum_{\substack{v \in A \\ u \in B}} \max\{m(u,v) - 1, 0\}$.

An algorithm is used that transforms D into a graph G, $G \subseteq K_{m,m}$, with no multiple edges, one where $M(G) = 0$. The algorithm does this transformation in a sequence of steps, replacing multiple edges with paths of length 3. In particular, the algorithm constructs a sequence of multigraphs $D_0 = D, D_1, \ldots, D_k = G$ in k stages such that all vertices of D_i have degree $\leqslant t + 2i$, $0 \leqslant i \leqslant k$, $M(D_j) < M(D_i)$ for $0 \leqslant i < j \leqslant k$, and $M(D_k) = 0$.

In the *ith* stage the algorithm transforms D_i to D_{i+1} using at least $m - 2t - 4i$ elementary steps, each step reducing the total multiplicity of D_i by one. To see this assume $u \in A$, $v \in B$, $m(u,v) \geqslant 2$, and that the transformation is in the *ith* stage with $d_{D_i}(u)$, $d_{D_i}(v) \leqslant t + 2i$. Let $S_A = \{y \in A \mid vy \notin E(D_i)\}$, $S_B = \{y \in B \mid uy \notin E(D_i)\}$, and let $z \in S_A$. Then $|S_A|$, $|S_B| \geqslant m - t - 2i$ so that z is nonadjacent to at least $m - 2t - 4i$ vertices in S_B. Let w be anyone of these $m - 2t - 4i$ vertices in S_B. An elementary step consists of replacing one of the multiple edges joining u and v by the 3-edge path $uwzv$.

As long as $m - 2t - 4i > 0$ and there are multiple edges in D_i the elementary step is repeated in stage i. Surely the *ith* stage has at least $m - 2t - 4i$ elementary steps and $M(D_i) < M(D_{i+1})$. Also another stage is possible as long as $i \leqslant (m - 2t)/4$ so $k \leqslant \lfloor (m - 2t)/4 \rfloor$. Since $M(D) \leqslant mt - m$, one needs $\sum_{i=0}^{\lfloor \frac{m-2t}{4} \rfloor} (m - 2t - 4i) \geqslant mt - m$. The reader can easily check that this last inequality holds when $t \leqslant m/12$ (and fails in general for $t \geqslant m/11$). □

One should recall that the only pairings considered in Section 2 for the graph $G(m,t)$ were ones determined by t-regular demand graphs with A and B as parts. Suppose one modifies $G(m,t)$ to a graph G^* so that the number of end vertices adjacent to each a_i (and b_i) may vary as long as the total number adjacent to A is identical to the total number adjacent to B. Further it is no longer required that the demand graphs which represent all possible pairings be regular. This suggests the following question for the modified graph G^*.

Question. Let $D(G^*)$ be a demand graph for G^* such that no edge has multiplicity greater than 2 and $|E(D(G^*))| \leqslant m^2/2$. What degree constraints must be placed on vertices of $D(G^*)$ so that the pairing given by $D(G^*)$ can be realized by edge disjoint paths of G^*?

If $|E(D(G^*))| = m^2/2$, then all edges of $D(G^*)$ are of multiplicity 2 and the pairing specified by $D(G^*)$ requires use of all edges of the $K_{m,m}$ (contained in G^*) to realize the edge disjoint paths of G^*. It is not difficult to see that when all edges of $D(G^*)$ have multiplicity 2 and $|E(D(G^*))| = m^2/2$ then $D(G^*)$ has severe degree restrictions. Clearly no vertex is of degree greater then m (counting multiplicities). Also one can show that there exist multigraphs $D(G^*)$ with all edges of multiplicity 2 and maximum degree $\leqslant 7m/9$ for which the required pairing fails.

The first interesting unsettled case in when $D(G^*)$ is regular of degree $m/2$ with each edge of multiplicity 2 and with $|E(D(G^*))| = m^2/2$. In this special case, one can reformulate the question as follows. Let m be a multiple of 4. Can one take an arbitrary $m/4$ regular spanning subgraph H of $K_{m,m}$ and decompose $K_{m,m}$ into edge disjoint 4-cycles such that each edge of H appears in a different 4-cycle? The existence of such arbitrary factorizations of $K_{m,m}$ into 4-cycles would in itself be of interest. For $m = 8$ and $m = 12$ the answer is affirmative.

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