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A VARIANT OF THE CLASSICAL RAMSEY PROBLEM

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For fixed integers p,q an edge coloring of a complete graph K is called a (p,q)-coloring if the edges of every $K_p \subseteq K$ are colored with at least q distinct colors. Clearly, $(p,2)$-colorings are the classical Ramsey colorings without monochromatic K_p subgraphs. Let $f(n,p,q)$ be the minimum number of colors needed for a (p,q)-coloring of K_n. We use the Local Lemma to give a general upper bound for f. We determine for every p the smallest q for which $f(n,p,q)$ is linear in n and the smallest q for which $f(n,p,q)$ is quadratic in n. We show that certain special cases of the problem closely relate to Turán type hypergraph problems introduced by Brown, Erdős and T.Sós. Other cases lead to problems concerning proper edge colorings of complete graphs.

1. Introduction

This note gives some results and problems concerning a question asked by the senior author in 1981. After some preliminary results of the senior author with Gy. Elekes and Z. Füredi (stated in [3], section 9), the area has been abandoned for fifteen years. Elekes went as far as claiming that the previous work has been done with another Elekes. The results claimed in [3] will be indicated by $(*)$ in this note.

An edge coloring of a complete graph K_n is a (p,q)-coloring if every subset of p vertices spans at least q distinct colors. It will be always assumed that $3 \le p$ and $2 \le q \le \binom{p}{2}$. The cited problem from [3] is: what is the least number of colors needed for a (p,q)-coloring of K_n? This number is denoted by $f(n,p,q)$ and we restrict our attention to the case when p,q are fixed and n tends to infinity. (We use f in an equivalent but slightly different way that in [3].) There is one trivial case to exclude: $f(n,p,\binom{p}{2}) = \binom{n}{2}$ if $p>3$.

Determining $f(n,p,q)$ for small values of p,q leads to problems of varying difficulty. Observe that $(p,2)$-colorings are equivalent to colorings without monochromatic K_p. Therefore determining $f(n,p,2)$ is hopeless, being equivalent to determining classical Ramsey numbers for multicolorings. For example, $R(3,3,3)=17$ can be expressed in terms of f as $f(16,3,2)=3, f(17,3,2)=4$. Thus the order of magnitude of $f(n,3,2)$ is equivalent to a well known open problem, classical results

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give $c_1\frac{log(n)}{loglog(n)} \le f(n,3,2) \le c_2 log(n)$. On the other hand, (3,3) colorings are equivalent with proper edge colorings, so $f(n,3,3)$ is the chromatic index of K_n (n for odd n, $n-1$ for even n). The most annoying problem among the small cases is the behavior of $f(n,4,3)$: the $o(n)$ upper bound (*) is improved here to $c\sqrt{n}$ (a special case of Theorem 1), but we could not even prove that $\frac{f(n,4,3)}{log(n)}$ tends to infinity. To show that $f(9,4,3)=3$ and $f(10,4,3)=4$ is not a difficult exercise, but to find the smallest n for which $f(n,4,3)=5$ is probably not so easy. (That number is smaller than 18, this follows from $R(4)=18$. To gain one, one has to show that K_{17} can not be decomposed into four edge disjoint graphs such that the union of any two is isomorphic to the (unique) Ramsey-graph on 17 vertices.)

In Section 2 a nonconstructive upper bound is given for $f(n,p,q)$. Sections 3-7 discuss problems and results for general p and specific q to clarify transitions of f like from sublinear to linear, from linear to superlinear, etc. The smallest q for which $f(n,p,q)$ is linear in n equals to $\binom{p}{2}-p+3$ (Theorem 2.). The smallest q for which $f(n,p,q)$ is quadratic in n equals to $\binom{p}{2}-\lfloor\frac{p}{2}\rfloor+2$ if $p\ge 4$ (Theorem 4.).

Concerning small values of p and q, we show that $\frac{5(n-1)}{6} \le f(n,4,5) \le n$ (Theorem 3.). A well-known result of Ruzsa and Szemerédi is used to show that $f(n,9,34) = \binom{n}{2} - o(n^2)$ (Theorem 5.). We think that the most interesting open problems are to find the order of magnitude for $f(n,4,4)$ and $f(n,5,9)$, at least to show that the latter is nonlinear.

2. Bound from Local Lemma

Theorem 1. $f(n,p,q) \le cn^{\frac{p-2}{\binom{p}{2}-q+1}}$, *where c depends only on p,q.*

Proof. Consider a random t-coloring of the edges of K_n, each edge gets color i with probability $\frac{1}{t}$. The probability that p fixed vertices span at most $q-1$ colors is bounded by

$$P = t^{q-1}\left(\frac{q-1}{t}\right)^{\binom{p}{2}}$$

where t^{q-1} is the overestimate of the number of ways to select at most $q-1$ colors from t colors. Since colorings of p element sets which intersect in at most one vertex are independent, the Local Lemma ([6]) can be applied with dependency bound $D=\binom{p}{2}\binom{n}{p-2}$. Now the theorem follows from solving $ePD<1$ for t. ∎

3. Limits for linear $f(n,p,q)$

For fixed p, it is easy to find the smallest q for which $f(n,p,q)$ is linear in n. This is given in the next theorem.

Theorem 2. *For arbitrary p, set $q=\binom{p}{2}-p+3$. Then $f(n,p,q)$ is linear in n while $f(n,p,q-1)$ is sublinear ($\leq cn^{1-\frac{1}{p-1}}$).*

Proof. The upper bounds follow from Theorem 1. Consider a (p,q)-coloring of K_n (with the value of q stated in the theorem). The graph determined by the edges of an arbitrary color class has maximum degree at most $p-2$, otherwise we have a vertex incident to $p-1$ edges of the same color which determines p vertices spanning at most $\binom{p}{2}-p+2=q-1$ colors, contradicting to the definition of the (p,q)-coloring. Therefore each color class has at most $\frac{(p-2)n}{2}$ edges. This implies that one needs at least $\frac{n-1}{p-2}$ colors for a (p,q)-coloring of K_n. ∎

Since $f(n,4,6)=\binom{n}{2}$ is trivial, Theorem 2 gives that $f(n,4,q)$ is linear if and only if $q=5$. However, it seems to be interesting to find the asymptotics. The lower bound of the next theorem is claimed in (*).

Theorem 3. *$\frac{5(n-1)}{6} \leq f(n,4,5)$ for all $n \geq 4$. For all odd n, $f(n,4,5) \leq n$, for infinitely many even n, $f(n,4,5) \leq n-1$.*

Proof. Observe that in a $(4,5)$-coloring of K_n each color class is the union of vertex disjoint edges and paths of length two. Assume that there are m color classes, and in color class i there are t_i edges and h_i paths of length two. Clearly, for every i, $2t_i+3h_i \leq n$, therefore

$$2\sum_{i=1}^{m} t_i + 3\sum_{i=1}^{m} h_i \leq mn.$$

Another property of a $(4,5)$-coloring is that if $x_1x_2x_3$ is a 2-path in color i and the edge x_1x_3 is colored with color j $(i \neq j)$, then x_1x_3 is an edge component in color class j. This implies that the number of 2-path components can not be larger than the number of edge components:

$$\sum_{i=1}^{m} t_i \geq \sum_{i=1}^{m} h_i$$

Finally, since each edge of K_n is colored, we have

$$\sum_{i=1}^{m} t_i + 2\sum_{i=1}^{m} h_i = \binom{n}{2}.$$

The claimed lower bound follows easily from the displayed inequalities.

To get the stated upper bound for odd n, consider the standard rotational decomposition of K_n into n factors, each of them spanning all but one vertices. In this decomposition there is no alternating C_4 (cycle on four vertices). This ensures that the coloring is a $(4,5)$-coloring. The standard extension of this coloring to even n gives the required $(4,5)$-coloring with $n-1$ colors whenever there is no alternating C_4 in the union of any two color classes. This certainly happens if $n-1$ is a prime which is enough for our purposes. (The condition that there is no C_4 in the union of any two color classes in a factorization of K_n (n is even) is a weakening of the notion of *perfect factorization* where the union of any two color classes must form a Hamiltonian cycle, see [11]. It is an open problem whether a perfect factorization exists for all even n. Perhaps it is known that for all even n there is a factorization with this weaker property.) ∎

The authors disagree in predicting the truth about $f(n,4,5)$. The senior author feels that $n-o(n)$ is the truth which means that only a negligable number of 2-paths can be used in a $(4,5)$-coloring of K_n. The junior author thinks that $\frac{5n}{6}$ is closer to the true value. The problem is probably not trivial, the values of f for small n are: 5 for $n=4,5,6$; 7 for $n=7,8$. Up to this point extremal $(4,5)$-colorings can be provided from the standard factorization ($n=4$ is a trivial degenerate case). The case $n=9$ is interesting, $f(9,4,5)=8$ and in the extremal $(4,5)$-coloring we found, six color classes are in the form P_3+3K_2 and two in the form $3K_2$. We describe it with the convention that ij is an edge and ijk is the 2-path with edges ij, jk. Of course, it would be interesting to generalize this factorization.

A $(4,5)$-coloring of K_9 with eight colors:

123, 49, 57, 68; 564, 18, 29, 37; 879, 16, 24, 35; 417, 26, 38, 59;
258, 19, 34, 67; 396, 15, 27, 48; 28, 36, 47; 13, 45, 89;

The upper limit of linearity seems to be a difficult but interesting problem. More precisely, the question is that for a fixed p what is the largest q (in terms of p) such that $f(n,p,q)$ is linear in n? The difficulty appears already for $p=5$ which is discussed next.

Recall that (from Theorem 2) $f(n,5,8)$ is linear (but $f(n,5,7)$ is sublinear). What about $f(n,5,9)$? Consider a $(5,9)$-coloring of K_n. Call a color class proper if it has pairwise disjoint edges. It is immediate to check that a non-proper color class has precisely two edges, forming a 2-path. Moreover the 2-paths formed by distinct non-proper color classes are pairwise vertex disjoint. Therefore at most $\frac{n}{3}$ color classes are non-proper, i.e. they do not affect the linearity of f. In fact, one can introduce at most $\frac{n}{3}$ new colors to get a proper $(5,9)$-coloring from the given $(5,9)$-coloring. The crucial point is that the union of two proper color classes can not form a path (or a cycle) with four edges. Thus the linearity of $f(n,5,9)$ is equivalent with the following problem.

Problem 1. Is it possible to have a proper edge coloring of K_n with cn colors so that the union of any two color classes has no paths or cycles with four edges?

The answer is probably negative and might come from Szemerédi lemma. (Perhaphs only cn^2 edges are needed instead of $\binom{n}{2}$.)

Problem 2. Decide whether for every p, $f(n,p,q)$ is linear only if $q=\binom{p}{2}-p+3$.

4. Quadratic $f(n,p,q)$

One can easily find the value of q where f becomes quadratic.

Theorem 4. *Set $q=\binom{p}{2}-\lfloor\frac{p}{2}\rfloor+2$. Then $\binom{n}{2}\left(\lfloor\frac{p}{2}\rfloor-1\right)^{-1}\leq f(n,p,q)$. On the other hand, $f(n,p,q-1)\leq cn^{2-\frac{4}{p}}$. Moreover, for $p\geq 7$, $n^{\frac{4}{3}}-n^{\frac{2}{3}}\leq f(n,p,q-1)$.*

Proof. The first part follows from the observation that in a (p,q)-coloring of K_n each color class has at most $\lfloor\frac{p}{2}\rfloor-1$ edges. The upper bound for the second part comes from Theorem 1.

The lower bound comes from the following argument. If all color classes in a (p,q)-coloring of K_n have at most $\frac{n^{\frac{2}{3}}}{2}$ edges then at least $n^{\frac{4}{3}}$ colors must be used. Otherwise a color class C contains at least $e=\frac{n^{\frac{2}{3}}}{2}$ edges. From the choice of q and from the assumption $p\geq 5$, the edges in C are pairwise independent (otherwise, since n is large, we could select p vertices which span more than $\lfloor\frac{p}{2}\rfloor$ edges of C). The same argument and the assumption $p\geq 7$ gives that no color other than C is repeated in the subgraph spanned by the edges of C. Therefore one needs at least $\binom{2e}{2}-e=n^{\frac{4}{3}}-n^{\frac{2}{3}}$ colors in this case. ∎

5. When do we have $f(n,p,q)=\frac{n^2}{2}-o(n^2)$?

The exact value of q (in terms of p) for this transition seems to be difficult to find. This problem is related to a well-known problem of Brown, Erdős and T.Sós ([1]): determine the minimum number of edges of an r-uniform hypergraph on n vertices which ensures that there exist k vertices spanning at least s edges. Special cases of this problem appeared in [4],[2]. The problem whether the minimum is $o(n^2)$ for $r=3,k=6,s=3$ was asked in [1] and answered affirmatively by Ruzsa and Szemerédi in [10].

The probabilistic construction of Brown, Erdős and Sós (Theorem on p. 59 in [1]) gives the following.

Proposition. *Assume that p is even and set $q=\binom{p}{2}-\frac{p}{2}+2$. Then $f(n,p,q)\leq(\frac{1}{2}-\varepsilon)n^2$ where ε is positive and depends only on p.*

Proof. The special case ($r=4$) of the theorem proved in [1] (p. 59) shows that there is a 4-uniform hypergraph on n vertices and with $cn^{\frac{4s-k}{s-1}}$ edges ($c>0$) such that each k-element subset of vertices spans at most $s-1$ edges. We set $k=p$ and $s-1=\frac{p}{2}-2$. With this choice, there are at least cn^2 hyperedges. The property of the hypergraph implies that at most $\frac{p}{2}-2$ edges contain any given pair of vertices. Therefore one can keep a positive percentage, say $\frac{\varepsilon}{2}n^2$ edges such that no two of them intersect in more than one vertex. Let H be the hypergraph determined by these edges, assume its vertex set is V, $|V|=n$. Associate an edge coloring of K_n to H as follows. Divide each edge of H into two pairs and color both pairs with the same color. For distinct edges of H distinct colors are used. This gives a partial coloring of the pairs of V which is extended by coloring all uncolored pairs of V with distinct new colors. From the definition of H, this coloring of K_n is a (p,q)-coloring using $\binom{n}{2}-\varepsilon n^2$ colors.

Remark. For even p, Theorem 4 and Proposition 1 show that when $f(n,p,q)$ becomes quadratic in n, the coefficient of n^2 is less than $\frac{1}{2}$.

The connection of $f(n,p,q)$ with the Ruzsa-Szemerédi theorem ([10]) appears first for $p=9$ and illustrated in the next theorem.

Theorem 5. $f(n,9,\binom{9}{2}-2)=\binom{n}{2}-o(n^2)$.

Proof. Consider a $(9,q)$-coloring of K_n where $q=\binom{9}{2}-2$. Assume that m colors are used in this coloring. A color class is called single if it has only one edge, and called a twin if it has two disjoint edges. We claim that the union of all color classes which are not single or twin has at most cn edges.

To prove the claim, observe that a color class which is not single or twin contains either two intersecting edges (A) or three disjoint edges (B). Moreover, each color class has at most three edges from the definition of the $(9,q)$-coloring. At most $\frac{n}{3}$ color classes can have A-configurations and at most $\frac{n}{6}$ color classes can have B-configurations since these color classes span pairwise disjoint triples and six-tuples. From these remarks the claim follows.

The crucial claim is that we have $o(n^2)$ twin color classes. Consider the 4-uniform hypergraph H defined on the vertices of K_n with edges determined by the twin color classes. It is possible that H has edges of multiplicity two but for the proof we may discard multiple edges, keeping a positive fraction of edges for H. Moreover, using the well-known theorem of Erdős and Kleitman ([5]) we can assume that H is 4-partite (keeping a $\frac{4!}{4^4}$ fraction of edges). Also we may assume that H has minimum degree 3 because deleting edges incident to a vertex of degree at most two is a loss of at most cn edges. Observe that now two edges of H intersect in at most one vertex otherwise we have three edges of H whose union covers at most 9 vertices and this contradicts the fact that we have a $(9,\binom{9}{2}-2)$ coloring. Let X be a vertex class of the 4-partite hypergraph H and consider the 3-uniform hypergraph H^* which is defined by the removal of X from the vertex set of H and from all edges of H. Then H^* has the same number of edges as H has. Moreover, any six vertices of H^* carry at most two edges of H^*: if not then e_i for $i=1,2,3$ cover six vertices of H^* and their extension with $x_i \in X$ gives three edges of H covering at most 9 vertices, contradicting the fact that we have a $(9, \binom{9}{2}-2)$ coloring. Now the celebrated theorem of Ruzsa and Szemerédi ([10]) implies that H^* (thus H) has $o(n^2)$ edges and the claim is proved.

We conclude that after the removal of $o(n^2)$ edges all color classes are single therefore $m \geq \binom{n}{2} - o(n^2)$. ∎

The proof of Theorem 5 suggests that it might be difficult to find for general p the largest value of q for which $f(n,p,q) \leq (\frac{1}{2}-\varepsilon)n^2$ for some $\varepsilon > 0$ depending on p. It is not difficult to prove that

$$f\left(n, p, \binom{p}{2} - \left\lfloor \frac{p-4}{3} \right\rfloor\right) = \binom{n}{2} - o(n^2).$$

6. When do we have $f(n,p,q) = \binom{n}{2} - c$?

The next result gives for each p the smallest q such that all but a constant (depending on p) number of edges of K_n must be colored with distinct colors in a (p,q)- coloring of K_n.

Theorem 6. *Set* $q = \binom{p}{2} - \lfloor \frac{p}{4} \rfloor + 1$. *Then* $f(n,p,q) \geq \binom{n}{2} - 2\lfloor \frac{p}{4} \rfloor$ *but* $f(n,p,q-1) \leq \binom{n}{2} - \frac{n}{4}$.

Proof. Consider a (p,q)-coloring of K_n where q is defined in the theorem. Define a hypergraph on the vertices of K_n by repeating the following procedure. Select two edges of the same color and define a hyperedge by the 3 or 4 vertices spanned by these edges. Delete the two edges used. At most $\lfloor \frac{p}{4} \rfloor - 1$ hyperedges are defined, otherwise their union is contained in a p-element subset with at most $q-1$ colors. This gives the first part of the theorem.

For the second part define a $(p,q-1)$-coloring of K_n by taking $\lfloor \frac{n}{4} \rfloor$ disjoint 4-element subsets on $V(K_n)$ and in each of these subsets color two disjoint edges with color i (for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$), all other edges are colored with distinct new colors. ∎

7. When do we have $f(n,p,q) \geq n^{\varepsilon}$?

Theorem 7. *For* $p \geq 3$, $n^{\frac{1}{p-2}} - 1 \leq f(n,p,p) \leq cn^{\frac{2}{p-1}}$ *where c depends only on p.*

Proof. The upper bound comes from Theorem 1. The lower bound is proved by induction on p. The case $p=3$ is obvious. Assume that $p > 3$, K_n is given with a (p,p)-coloring with less than $t=n^{\frac{1}{p-2}}-1$ colors. This implies that some vertex of K_n is adjacent in the same color with a set A of at least $\frac{2\binom{n}{2}}{nt}=\frac{n-1}{nt}\geq n^{\frac{p-3}{p-2}}$ elements. The restriction of the coloring to A is clearly a $(p-1,p-1)$-coloring thus, by induction

$$f(n,p,p)\geq |A|^{\frac{1}{p-3}}-1\geq \left(n\frac{p-3}{p-2}\right)^{\frac{1}{p-3}}-1=n^{\frac{1}{p-2}}-1$$

which finishes the proof. ∎

Problem 3. Decide whether $f(n,p,p-1)\geq c(p)n^{\varepsilon(p)}$. A more general problem is to find (or estimate) the range of q (in terms of p) where $f(n,p,q)\geq c(p)n^{\varepsilon(p)}$. It is easy to see that $f(n,p,\lceil log(p)\rceil)\leq \lceil log(n)\rceil$ with base 2 logarithm.

8. Problems for small values of p,q

The problem of $f(n,4,3)$ is already mentioned in the Introduction. It would be desirable to decrease the probabilistic upper bound $c\sqrt{n}$. Concerning $f(n,4,4)$, it is already noted in (*) that $c\sqrt{n}\leq f(n,4,4)$ because a color class in a $(4,4)$-coloring of K_n can not contain a cycle of length four. (The same lower bound comes from Theorem 7.) The upper bound $cn^{\frac{2}{3}}$ follows from Theorem 1. Since $f(n,4,5)$ is linear, the problem is the coefficient of n (between $\frac{5}{6}$ and 1). For $p=5$ perhaps the most interesting problem is to decide whether $f(n,5,9)$ is linear (see Problem 1). There is a significant jump from $f(n,5,6)$ to $f(n,5,7)$ because the former is at most $cn^{\frac{3}{5}}$ (Theorem 1) and it is not difficult to see that the latter is at least $cn^{\frac{2}{3}}$. The behavior of $f(n,p,\binom{p}{2}-1)$ can be easily determined for $p>5$: $f(n,6,14)=\frac{5n^2}{12}+O(n)$, $f(n,7,20)=\binom{n}{2}-\lfloor\frac{n}{4}\rfloor$ and for $p>7$, $f(n,p,\binom{p}{2}-1)=\binom{n}{2}-1$.

9. Geometry

One gets difficult geometric problems if the vertices of K_n are points in d-dimensional space and colors of the edges are defined by distances, two edges have the same color if and only if the corresponding distances are the same. These colorings can be called distance colorings. Then $f_d(n,p,q)$ can be defined as the minimum number of colors over all (p,q) distance colorings determined by n points of R^d. Clearly, $f_d(n,p,q)\geq f(n,p,q)$ for every d. The senior author and Vera T.Sós proved that $f_1(n,4,5)\geq\binom{n}{2}-n+2$ ([7]). In fact, this is sharp for $n\geq 8$. Information about $(4,5)$ distance colorings in one dimension is given in [9]. It seems that nothing is known about dimension two, neither is known whether $\frac{f_2(n,4,5)}{n} \to \infty$ or $f_2(n,4,5)=o(n^2)$. Fishburn ([8]) has an example showing $f_2(5,4,5)=f(5,4,5)=5$.

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References

[1] W. G. Brown, P. Erdős, V. T. Sós: Some extremal problems on r-graphs, in *New directions in the theory of Graphs, Proc. 3rd Ann Arbor Conference on Graph Theory,* 53–63, Academic Press, New York, 1973.

[2] W. G. Brown, P. Erdős, V. T. Sós: On the existence of triangulated spheres in 3-graphs and related problems, *Periodica Mathematica Hungarica*, **3** (1973), 221–228.

[3] P. Erdős: Solved and unsolved problems in combinatorics and combinatorial number theory, *Congressus Numerantium,* **32** (1981), 49–62.

[4] P. Erdős: Extremal problems in graph theory, in *Theory of Graphs and its Applications* (M.Fiedler, ed.) Academic Press, New York, 1964, 29–36.

[5] P. Erdős, D. J.Kleitman: On coloring graphs to maximize the proportion of multicolored k-edges, *J. of Combinatorial Theory*, **5** (1968), 164–169.

[6] P. Erdős, L.Lovász: Problems and results on 3-chromatic hypergraphs and some related problems, in: *Infinite and Finite sets, Colloquia Math. Soc. J. Bolyai* vol. 10 (A.Hajnal et al eds.) North-Holland, Amsterdam, 609–617.

[7] P. Erdős, V. T. Sós: personal communication, 1994.

[8] P. C. Fishburn: personal communication, 1995.

[9] A. Gyárfás, J. Lehel: Linear Sets with Five Distinct Differences among any Four Elements, *J. of Combinatorial Theory B,* **64** (1995), 108–118.

[10] I. Z. Ruzsa, E. Szemerédi: Triple systems with no six points carrying three triangles, in *Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18,* Volume II. 939–945.

[11] W. D. Wallis: Combinatorial Designs, Monographs and Textbooks in *Pure and Applied Mathematics,* Vol. 118, Marcel Dekker, Inc. 1988.

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