Nonsymmetric Party Problems

András Gyárfás*
COMPUTER AND AUTOMATION INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES

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Abstract: The classical Ramsey problem is considered for antisymmetric digraphs and for tournaments. Three small Ramsey-type numbers are determined with some remarks concerning the general case. © 1998 John Wiley & Sons, Inc. J Graph Theory 28: 43–47, 1998

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1. INTRODUCTION

Ramsey numbers are usually introduced by the statement that among six people there are always three who either pairwise know each other or pairwise do not know each other. This statement assumes that the relation ‘‘to know each other’’ is symmetric. This note explores the rather realistic possibility of having nonsymmetric acquaintances. In this case we can claim that among nine people either there are three knowing each other transitively (A knows B and C, B knows C) or there are three who pairwise do not know each other. The claim is not true for eight people (like the symmetric version for five people).

Let $R(p, q)$ denote the classical Ramsey number, the smallest n for which every graph of order n contains either a complete subgraph of p vertices or an independent set of q vertices. The nonsymmetric analogue, $R^*(p, q)$, is defined as the smallest n for which every *digraph* of order n contains either a transitive complete subgraph of

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p vertices or an independent set of q vertices. Obviously we can restrict ourselves to antisymmetric digraphs, i.e., between any two vertices there is at most one directed edge. Unlike $R(p,q)$, the function $R^*(p,q)$ is not symmetric, $R^*(2,q) = q$ (trivial) but $R^*(p,2)$ is the smallest n for which every tournament T_n contains a transitive subtournament T_p. It is well-known that

$$2^{\frac{p-1}{2}} \leq R^*(p,2) \leq 2^{p-1}$$

and the lower bound comes from the probability method (see Ref. [4]) while the upper bound is easy by induction (see in [7]). It is known that $R^*(4,2) = 8$ and $R^*(5,2) = 14$. The first new small value of $R^*(p,q)$ comes easily.

Theorem 1. $R^*(3,3) = 9$.

Proof. $R^*(3,3) \leq 9$ follows immediately from $R(4,3) = 9$ by noting that any tournament T_4 contains a transitive T_3. The other direction follows by observing that the (Ramsey-)graph which is obtained from C_8 by adding the short diagonals can be oriented so that all triangles are cyclic. In fact, it is easy to check that this is the only digraph on eight vertices without transitive triangles and without three independent vertices. ∎

The well-known recursive bound $R(p,q) \leq R(p-1,q) + R(p,q-1)$ is paralleled in the next proposition.

Proposition 1. $R^*(p,q) \leq 2R^*(p-1,q) + R^*(p,q-1) - 1$.

Proof. Assume that a digraph D has no transitive K_p and has no independent set of size q. From an arbitrary vertex x of D one can partition $V(D) - \{x\}$ into $A \cup B \cup C$, where A and B are the outset and inset of x and C is the set of vertices nonadjacent with x. Using induction, both $|A|$ and $|B|$ are at most $R^*(p-1,q) - 1$ and $|C| < R^*(p,q-1)$. Therefore

$$|V(D)| \leq 1 + 2R^*(p-1,q) - 2 + R^*(p,q-1) - 1$$
$$= 2R^*(p-1,q) + R^*(p,q-1) - 2$$

which gives the required inequality. ∎

Before determining the value of $R^*(3,4)$, it is shown that

$$14 \leq R^*(3,4) \leq 16.$$

The digraph for the lower bound is found by a former Budapest Semesters student, Kesten Smith. Using the notation $|n|$ for $\{1,2,\ldots,n\}$, take the 3-regular digraph on [13] where vertex i has the outset $\{i+1, i+4, i+11\}$ (mod 13). The upper bound follows from Proposition 1 and Theorem 1 since $R^*(3,4) \leq 2R^*(2,4)| + R^*(3,3) - 1 = 8 + 9 - 1 = 16$.

Theorem 2. $R^*(3,4) = 15$.

Proof. The upper bound is proved first. Let D be a digraph of order 15. Since a vertex of in- or outdegree at least four or a vertex with nine nonadjacencies finishes the proof, assume that D is a 3-regular digraph. Moreover, from the remark at the end of the proof of Theorem 1, the eight vertices nonadjacent to a given vertex of D induce a 2-regular digraph isomorphic to the digraph on vertex set [8] in which the outset of i is $\{i+1, i+6\}$ (mod 8). For an arbitrary vertex x, let A_1, A_2 denote the outset and inset of x and set $C = V(D) \setminus (A_1 \cup A_2 \cup \{x\}), A = A_1 \cup A_2$. Let B denote the bipartite graph with vertex classes A, C in which ac is an edge if and only if ac or ca is edge of D. Clearly, A_1 and A_2 are independent and edges between A_1 and A_2 are oriented from A_1 to A_2. As stated before, C induces a 2-regular subdigraph in D, therefore B has 16 edges. Moreover, there are precisely eight edges between A_i and C for $i = 1, 2$ otherwise A_1 or A_2 can be extended with a vertex of C to a four-element independent set. Each $x \in A$ has degree at least two in B and this easily implies that either A_1 or A_2, say A_1 has degree sequence 2, 3, 3 in B. This means that (in B) three vertices of C are adjacent to a vertex of A_1 but nonadjacent to the other two vertices of A_1. This condition implies that D contains K_4 or four independent vertices.

The lower bound comes from the following 3-regular digraph D. The vertex set of D is [14] and the outset of i is defined as $\{i+1, i+4, i+12\}$ for even i and as $\{i+1, i+8, i+12\}$ for odd i (mod 14). Due to the symmetry of D, it is easy to check that each triangle of D is cyclic and there are at most three independent vertices in D. In fact one has to check (for even and odd i) that all edges between the outset and inset of i are oriented properly and the deletion of i and its neighbors leaves a graph H with no three independent vertices. The complement of H is isomorphic to either C_7 with diagonals 14, 25 or to C_7 with diagonals 14, 26 (depending the parity of i). ∎

It is probably difficult to determine any other value of $R^*(p, q)$ if both parameters are at least three. For the diagonal case, one can parallel the well-known Ramsey bounds as follows. (The upper bound is weaker than the one coming from Proposition 1.)

Proposition 2. $3^{\frac{p-1}{2}} \leq R^*(p, p) \leq 3^{2p-2}$.

Proof. The lower bound comes by repeating the classic probabilistic proof of Erdős to a random antisymmetric digraph. The upper bound also follows the well-known majority argument: Starting from an arbitrary vertex, select the largest from the inset, outset, nonadjacency-set and repeat the procedure. At least $2p - 1$ steps can be done. If nonadjacency sets were selected p times we have p independent vertices. Otherwise adjacencies (in or out) were selected p times and this gives a transitive T_p. ∎

Using recent results of Kim (Ref. [6]) and Alon (Ref. [1]), the order of magnitude of $R^*(3, q)$ can be determined (it is the same as of $R(3, q)$). The work of Alon generalizes former results Ajtai-Komlós-Szemerédi (Ref. [2]) and Shearer (Ref. [8]).

Proposition 3. $\frac{c_1 q^2}{\log q} \leq R^*(3, q) \leq \frac{c_2 q^2}{\log q}$.

Proof. The lower bound is the consequence of the celebrated result of Kim [6] which bounds $R(3,q)$ from below by $\frac{cq^2}{\log q}$. For the upper bound, let D be a digraph with n vertices, without transitive triangles and without q independent vertices. Noting that the outset and inset of any vertex is independent, it follows that the underlying graph G of D has maximum degree at most $2q-1$ and the neighborhood of any vertex is bipartite. It follows from a result of Alon ([1, Theorem 1.1]) that G has an independent set of size at least $\frac{cn \log q}{q} < q$ which gives the required inequality. ∎

In the definition of R^* only the acquintances are directed. One might wonder to ask a symmetric version. This leads to define $f(p,q)$ as the smallest n for which in any red-blue coloring of a directed *symmetric* complete K_n there is a either a red transitive K_p or a blue transitive K_q. However, it is easy to see that $f(p,q) = R(p,q)$, this is proved by Harary and Hell in Ref. [5]. On the other hand, if $R^{**}(p,q)$ is the smallest n such that in any red-blue coloring of any *tournament* T_n there is either a red transitive T_p or a blue transitive T_q one gets a (seemingly) new function.

Proposition 4. $R^{**}(3,3) = 14$.

Proof. The lower bound comes from the following red-blue T_{13} with vertex set [13]. The red outset of i is $\{i+1, i+4, i+11\}$ and the blue outset of i is $\{i+3, i+5, i+7\}$ (mod 13). The upper bound follows noting that at any vertex x of a red-blue T_{14} there are four edges whose colors and directions (with respect to x) are the same. This gives a monochromatic T_4 which contains a transitive T_3. ∎

Perhaps it is very difficult to find any other nontrivial value of R^{**}. But there can be nice constructions which improve straightforward lower bounds like $22 \le R^{**}(3,4)$ (replacing each vertex of a red cyclic triangle by a blue Paley tournament). By a similar replacement, the two-colored tournament T_{13} from the proof of Proposition 4 can be used to get a three-colored T_{39} without monochromatic transitive triangle. This shows $40 \le R^{**}(3,3,3)$ (while it is well-known that $R(3,3,3) = 17$).

Proposition 2 can be paralleled by

Proposition 5. $2^{p-1} \le R^{**}(p,p) \le 4^{2p-2}$.

In spite of the huge difference for small numbers, it is not clear how to separate R from R^{**}.

Problem. *Show that* $\frac{R^{**}(p,p)}{R(p,p)}$ *tends to infinity.*

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