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NOTE

Generalized Split Graphs and Ramsey Numbers

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A graph G is called a (p, q)-split graph if its vertex set can be partitioned into A, B so that the order of the largest independent set in A is at most p and the order of the largest complete subgraph in B is at most q. Applying a well-known theorem of Erdős and Rado for Δ-systems, it is shown that for fixed p, q, (p, q)-split graphs can be characterized by excluding a finite set of forbidden subgraphs, called (p, q)-split critical graphs. The order of the largest (p, q)-split critical graph, $f(p, q)$, relates to classical Ramsey numbers $R(s, t)$ through the inequalities

$$2F(F(R(p+2, q+2))) + 1 \geqslant f(p, q) \geqslant R(p+2, q+2) - 1$$

where $F(t)$ is the smallest number of t-element sets ensuring a $t+1$-element Δ-system. Apart from $f(1, 1) = 5$, all values of $f(p, q)$ are unknown. © 1998 Academic Press

Split graphs have been introduced by Földes and Hammer in [FH] as graphs whose vertices can be partitioned into a complete graph and an independent set. It was proved in [FH] that split graphs can be characterized by the exclusion of three induced subgraphs: C_4, $2K_2$, and C_5. (The same result is obtained independently in a slightly more general form in [GL] in the context of 2-track interval systems.) A natural generalization of split graphs have been considered in [EG]; here we shall use a special case of that definition. First some more or less standard terminology is summarized.

We consider finite undirected simple graphs $G = (V, E)$, where V, E are the vertex set and edge set of G, respectively. The numbers $|V|, |E|$ are called the order and the size of the graph G. For $A \subseteq V$, $G[A]$ denotes the subgraph of G induced by A, if G is clear from the context, $[A]$ will be used. As usual, $\alpha(G)$ denotes the order of the largest independent set of G

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and $\omega(G)$ denotes the order of the largest complete subgraph of G. The set $\{1, 2, ..., k\}$ is abbreviated as $[k]$.

A graph $G=(V, E)$ is called a (p, q)-*split graph* if V can be partitioned into sets A, B so that $\alpha(G[A]) \leqslant p$ and $\omega(G[B]) \leqslant q$. The partition $[A, B]$ will be called a (p, q)-split partition of G.

A graph G is called (p, q)-*split critical* if G is not a (p, q)-split graph but all proper induced subgraphs of G are (p, q)-split graphs. In a (p, q)-split critical graph G each vertex v defines a (p, q)-split partition $[A_v, B_v]$ on $V(G)\backslash v$ and there exists an α-*witness* , S_v, which is an independent set of $p+1$ vertices containing v and disjoint from B_v. Similarly, there exists an ω-*witness* , K_v, a complete subgraph of G with $q+1$ vertices which contains v and is disjoint from A_v.

Note that a graph G is (p, q)-split if and only if its complement G is (q, p)-split. A similar remark is valid for split critical graphs. In particular, (p, p)-split graphs and (p, p)-split critical graphs form self-complementary families of graphs.

In [EG] the minimum order of split critical graphs studied (under a more general definition). This paper is focused on the maximum order of a (p, q)-split critical graph. It is not clear a priori that the maximum is finite, the main result of the paper is the proof of the finiteness (Theorem 1). This allows us to define the function $f(p, q)$ as the *maximum number of vertices in a* (p, q)-*split critical graph*. It is worth mentioning that Theorem 1 is an existence theorem, giving an upper bound which is probably very far from the actual value of $f(p, q)$ which might be difficult to determine in general.

A more restricted family of graphs, the *perfect* (p, q)-split critical graphs is the same as the class of graphs studied by Kézdy *et al.* in [KSW]. They proved (Theorem 2.5 in [KSW]) that there are finitely many perfect (p, q)-split critical graphs. Theorem 1 is a more general result but the bound of Kézdy *et al.* for the maximum number of vertices of a perfect (p, q)-split critical graph is better than the general bound coming from Theorem 1.

The immediate corollary of Theorem 1 is that (p, q)-split graphs can be characterized by the exclusion of finitely many induced subgraphs (Corollary 1). An explicit description of these graphs (like C_4, $2K_2$, C_5 for (1, 1)-split critical graphs) is not expected since (as Proposition 2 will show) all Ramsey graphs are among them.

The easiest example of a (p, q)-split critical graph is the graph $(p+1)K_{q+1}$, i.e. $p+1$ vertex-disjoint copies of the complete graph K_{q+1}. This example is of minimum order as the next proposition shows.

PROPOSITION 1. *Graphs of order at most* $pq+p+q$ *are* (p, q)-*split graphs.*

Proof. Let G be a graph of order at most $pq+p+q$. Select the maximum number of vertex disjoint complete subgraphs of G, each with $q+1$ vertices. The vertices covered by these complete graphs span a subgraph of G with no independent set of size $p+1$ and the uncovered vertices span a subgraph of G with no complete subgraph of size $q+1$. Thus G is a (p, q)-split graph. ∎

To get much larger examples of (p, q)-split critical graphs, let $R(s, t)$ be the classical Ramsey number, the smallest integer N for which every graph G of order N satisfies either $\alpha(G) \geqslant s$ or $\omega(G) \geqslant t$. (s, t)-*Ramsey graphs* are the graphs G of order $R(s, t)-1$ for which $\alpha(G) < s$ and $\omega(G) < t$. Ramsey numbers (and Ramsey graphs) are known only for small values of s and t. It is known that the (3, 3)-, (3, 4)-, and (4, 4)-Ramsey graphs are unique (the first is the pentagon, the others can be found e.g. in [GRS]).

PROPOSITION 2. *$(p+2, q+2)$-Ramsey graphs are (p, q)-split critical.*

Proof. Let G be a $(p+2, q+2)$-Ramsey graph. Assume that G is (p, q)-split with split partiton A, B. Then one can add a new vertex to G adjacent to all vertices of B (and to no vertices of A). The resulting graph G^* has $R(p+2, q+2)$ vertices and $\alpha(G^*) \leqslant p+1$, $\omega(G^*) \leqslant q+1$. This contradicts the definition of the Ramsey number $R(p+2, q+2)$. Thus G is not a (p, q)-split graph. On the other hand, for each $v \in V(G)$ the sets A_v, B_v can be defined as the set of vertices non-adjacent, respectively adjacent to v. Since G is a Ramsey graph, $\alpha(G[A_v]) \leqslant p$, $\omega(G[B_v]) \leqslant q$ follows immediately. Therefore G is (p, q)-split critical. ∎

It is tempting to conjecture that the Ramsey graphs are the largest split critical graphs. For (1, 1)-split critical graphs this follows from the split graph characterization theorem cited above. However, for $p=1$, $q=2$, the (3, 4)-Ramsey graph has eight vertices ($R(3, 4)=9$) but the graph obtained from the regular 9-gon by adding three pairwise non-intersecting shortest diagonals is an example of a (1, 2)-split critical graph. In fact, it is not clear whether this is a largest (1, 2)-split critical graph (even the claim that there is a largest one seems to be nontrivial). The following graph G_{18} on 18 vertices also beats by one the famous (4, 4)-Ramsey graph. Let M denote the six-vertex graph obtained by joining a new vertex to two non-consecutive vertices of a five cycle. The graph M has two vertices with the same set of neighbors, call them special vertices. Then G_{18} is defined by arranging 18 vertices into a 3×6 matrix in which each column forms a triangle and each row is isomorphic to a copy of M arranged so that the six special vertices of the three copies occupy distinct columns.

PROPOSITION 3. *The graph G_{18} is a (2, 2)-split critical graph.*

Proof. Assume that $[A, B]$ is a split partition of G_{18}. Since $\omega([B]) \leqslant 2$, B has at most two vertices from each column of G_{18}. Therefore one can select a six element subset $C \subseteq A$ with all vertices of C from distinct columns. Now $\alpha([C]) \leqslant 2$ because $C \subseteq A$ and $\omega([C]) \leqslant 2$ because only the columns form triangles in G_{18}. This contradicts to $R(3, 3) = 6$.

Let v be a vertex of G_{18} and w be the special vertex in the column of v ($v = w$ is possible). Then A_v is defined by removing w from the row of w and $B_v = V(G_{18}) \setminus (A_v \cup \{v\})$. Since A_v is a five-cycle and B_v has two vertices from each column, (A_v, B_v) is a (2, 2)-split partition of $V(G_{18}) \setminus v$. ∎

In terms of $f(p, q)$, the preceeding remarks show that $f(1, 1) = 5$, $f(1, 2) \geqslant 9$, $f(2, 2) \geqslant 18$. It would be interesting to determine $f(2, 2)$; the antisymmetry of G_{18} suggests that there are much larger examples of (2, 2)-split critical graphs.

In the proof of Theorem 1 we shall refer to the diagonal case of a well-known theorem of Erdős and Rado ([ER]) on Δ-systems (later several other names, like star and sunflower were introduced for Δ-systems). A hypergraph (set system) with edges $e_1, e_2, ..., e_t$ is called a Δ-system, if any two distinct edges intersect in the same set K, i.e. $e_i \cap e_j = K$ for all $1 \leqslant i < j \leqslant t$. The set K is called the kernel and the sets $e_i \setminus K$ are the rays (or petals) of the Δ-system. The rank of a hypergraph is the cardinality of its largest edge. The well-known theorem of Erdős and Rado is generally stated for simple r-uniform hypergraphs but it is immediate to check that it remains true if only rank r is assumed and multiple edges are also allowed. With this consideration the diagonal case of the Erdős–Rado theorem ([ER]) can be stated as follows.

THEOREM A. *A (not necessarily simple) hypergraph of rank r with more than* $F(r) = r!(r)^r$ *edges contains a* Δ*-system with* $r+1$ *edges.*

THEOREM 1. *For any fixed pair of positive integers p, q there are finitely many (p, q)-split critical graphs.*

Proof. Assume that G is a (p, q)-split critical graph. Let A denote a subset of $V = V(G)$ such that $\alpha(G[A]) \leqslant p$ and $|A|$ is largest with this property. Set $V = [n]$, $A = [m]$ and, for convenience, $\bar{A} = V \setminus A$. We are going to show that m is bounded by a function $g(p, q)$. This will imply the theorem because the same argument can be applied to $\bar{G}$ to show that $|B| \leqslant g(q, p)$ for every $B \subseteq V$ such that $\omega(G[B]) \leqslant q$. Then, using a (p, q)-split partition $[A_v, B_v]$ of $V(G) \setminus v$ (with arbitrary $v \in V$) we obtain

$$|V(G)| \leqslant g(p, q) + g(q, p) + 1.$$

For each $i \in A$, consider a (p, q)-split partition $[A_i, B_i]$ of $V \backslash i$ such that $|A_i \backslash A|$ is as small as possible. Set

$$X_i = A_i \backslash A, \; Y_i = A \backslash A_i.$$

Since $Y_i \subseteq A$, $\alpha([Y_i]) \leqslant p$. On the other hand, $Y_i \backslash i \subseteq B_i$ therefore $\omega([Y_i \backslash i]) \leqslant q$. Thus, by Ramsey's theorem, $|Y_i| < R(p+1, q+2)$. Also, from the choice of A, $|X_i| \leqslant |Y_i|$. Using $p+2$ to get a more symmetric formula,

$$|X_i| \leqslant |Y_i| < R(p+2, q+2) = r.$$

The sets X_i and Y_i form hypergraphs with m edges and the rank of both hypergraphs is at most r. Set

$$g(p, q) = F(F(r))$$

where F is the Erdős–Rado function from Theorem A. The proof will be finished by proving the following claim.

Claim. $m \leqslant g(p, q)$. If the claim is not true then the definition of g implies that for some $I \subset [m]$, $|I| = r+1$, the hypergraphs $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are both Δ-systems with $r+1$ edges. By rearranging indices, we may assume that $I = [r+1]$. Let X, Y denote the kernels of these Δ-systems and let X_i^*, Y_i^* denote the petals, i.e. $X_i^* = X_i \backslash X$, $Y_i^* = Y_i \backslash Y$ for $i \in [r+1]$.

We prove first that there is a non-empty petal X_i^* for some $i \in [r+1]$. If this were not true then $X = X_i$ for all $i \in [r+1]$. Since $|Y| \leqslant r$, there exists $j \in [r+1]$ such that $j \notin Y$, therefore $j \in Y_j^*$. Now for any α-witness S_j, $S_j \cap Y = \emptyset$ therefore S_j can have non-empty intersection with at most $p+1$ sets Y_i. Since $r+1 > p+1$ (in fact, $r+1 >> p+1$), there is a $k \in [r+1]$ such that $S_j \cap Y_k = \emptyset$. Since $X_j = X_k (= X)$, this implies that

$$S_j \subseteq ((A \backslash Y_k) \cup X_j) = ((A \backslash Y_k) \cup X_k) = A_k,$$

contradicting to the assumption $\alpha([A_k]) \leqslant p$.

Based on the previous paragraph, we may assume that $X_1^* \neq \emptyset$. Consider the partition A_1^*, B_1^* of $V \backslash \{1\}$ where A_1^* is defined as

$$A_1^* = (A_1 \backslash X_1^*) \cup (B_1 \cap Y_1^*)$$

and $B_1^* = V \backslash (A_1^* \cup \{1\})$. We are going to show that $[A_1^*, B_1^*]$ is a (p, q)-split partition of $V \backslash \{1\}$, i.e. $\alpha([A_1^*]) \leqslant p$ and $\omega([B_1^*]) \leqslant q$. Indeed, if A_1^*

has an independent set S with $|S| = p+1$, from $S \cap Y = \varnothing$, S can have non-empty intersection with at most $p+1$ Y_j sets. Therefore, for some $k \in [r+1]$, $S \cap Y_k = \varnothing$. But then $S \subseteq (A \backslash Y_k) \cup X \subseteq A_k$ contradicting $\alpha([A_k]) \leqslant p$. Assume that $B_1^* = (Y \backslash \{1\}) \cup (\bar{A} \backslash X)$ has a complete subgraph K with $|K| = q+1$. Select $k \in [r+1]$ such that $X_k^* \cap K$ and $\{k\} \cap K$ are both empty sets. Such a choice is possible, since $r+1 > q+1$, and in fact $r+1 >> q+1$. Now the vertices of K are covered by $(Y_k \backslash \{k\}) \cup (A \backslash X_k) \subseteq B_k$ which contradicts $\omega([B_k]) \leqslant q$. Therefore $[A_1^*, B_1^*]$ is a (p, q)-split partition of $V \backslash \{1\}$. This contradicts to the choice of A_1, because

$$|A_1^* \backslash A| = |A_1 \backslash (A \cup X_1^*)| = |X_1 \backslash X_1^*| < |X_1| = |A_1 \backslash A|$$

This final contradiction finishes the proof of the claim and the proof of Theorem 1. ∎

Corollary 1. *For fixed p, q, (p, q)-split graphs can be characterized by the exclusion of finitely many forbidden subgraphs.*

Combining Proposition 2 and the actual upper bound of Theorem 1 leads to the following estimates on $f(p, q)$.

Corollary 2. $R(p+2, q+2) - 1 \leqslant f(p, q) \leqslant 2F(F(R(p+2, q+2))) + 1.$

Remarks. Theorem 1 and its proof remain true for hypergraphs of fixed rank. Therefore Corollary 1 is also true for hypergraphs but Proposition 2 (and the lower bound of Corollary 2) collapses. The upper bound of $f(p, q)$ can certainly be improved. For example, Imre Bárány noted [B] that with a slight modification of the proof of Theorem 1 the iteration of F can be avoided by doubling the inner function $R(p+2, q+2)$.

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