ELSEVIER

Discrete Mathematics 177 (1997) 99–122

DISCRETE MATHEMATICS

On-line 3-chromatic graphs — II Critical graphs[1]

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Received 22 February 1995; revised 23 May 1996

Abstract

On-line coloring of a graph is the following process. The graph is given vertex by vertex (with adjacencies to the previously given vertices) and for the actual vertex a color different from the colors of the neighbors must be irrevocably assigned. The on-line chromatic number of a graph G, $\chi^*(G)$ is the minimum number of colors needed to color on-line the vertices of G (when it is given in the worst order). A graph G is on-line k-critical if $\chi^*(G)=k$, but $\chi^*(G')<k$ for all proper induced subgraphs $G'\subset G$. We show that there are finitely many (51) connected on-line 4-critical graphs but infinitely many disconnected ones. This implies that the problem whether $\chi^*(G)\leqslant 3$ is polynomially solvable for connected graphs but leaves open whether this remains true without assuming connectivity. Using the structure descriptions of connected on-line 3-chromatic graphs we obtain *one* algorithm which colors all on-line 3-chromatic graphs with 4 colors. It is a tight result. This is a companion paper of [1] in which we analyze the structure of triangle-free on-line 3-chromatic graphs.

0. Introduction

The *on-line chromatic number* of a graph G, $\chi^*(G)$, has been defined in [4] as

$$\inf_A \chi_A(G),$$

where the infimum is over all on-line algorithms and $\chi_A(G)$ is the maximum number of colors used by A over all possible orderings of the vertices of G.

The on-line chromatic number of a graph G can be defined also through a two-person game. The *Drawer* gives the vertices of G in some order together with adjacencies to

[1] Research supported by OTKA grants 2570, 4271 and 14302.

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PII S0012-365X(96)00359-7

vertices already given. The *Painter* assigns a legal color to the current vertex given. The aim of Drawer is to force Painter to use as many colors as possible. The aim of Painter is to use as few colors as possible. The common optimal value is the on-line chromatic number of G.

Problems and results about the on-line chromatic number can be found in [1,4–6,8].

A graph G is called *on-line k-critical* if $\chi^*(G)=k$ but $\chi^*(G-x)<k$ for all $x\in V(G)$. It is trivial that K_2 is the only on-line 2-critical graph and it is easy to show that K_3 and P_4 are the only on-line 3-critical graphs (K_m stands for the complete graph and P_m for the path on m vertices and, for later use, C_m for the cycle on m vertices and $K_{m,m}$ for the complete bipartite graph on $2m$ vertices. If G and H are graphs then $G+H$ denotes the disjoint union of the two graphs and $2G=G+G$). As a consequence, on-line 2-chromatic graphs are rather trivial: a graph is on-line 2-chromatic if and only if each connected component is a complete bipartite graph. Throughout this paper *subgraph* will always mean *induced subgraph* (i.e. if we say G contains P_4 it means that P_4 is an induced subgraph of G).

The next step, the family of on-line 3-chromatic graphs seems to be very interesting. In [1], which is a companion paper to this one, it was shown that bipartite on-line 3-chromatic graphs can be characterized by the exclusion of finitely many (10) induced on-line 4-critical subgraphs. Furthermore, the same result was extended to triangle-free graphs (with 22 excluded induced subgraphs). The results of this paper imply a similar 'finite basis theorem' for all connected on-line 3-chromatic graphs. In this case, there are 51 forbidden induced subgraphs (Theorem 1). This result can be extended to a large family of disconnected graphs (with more forbidden induced subgraphs). In fact, the family of graphs where on-line 3-colorability is not characterized is rather special: each component is either the complement of C_6 (the six-vertex cycle) or a graph which is on-line 3-colorable by the greedy algorithm (see Theorem 3).

Using the obtained 'finite bases' of connected on-line 3-chromatic graphs and a basic statement from [1] about one class of such graphs we present a very simple on-line coloring algorithm (in Section 2) which can color all on-line 3-chromatic graphs with 4 colors (Theorem 2). This result is the best possible since it was shown in [5] that this cannot be done with 3 colors. The same result is proved independently (using a different algorithm) by Kolossa [7].

The authors believed that there is a 'finite basis theorem' for all on-line 3-chromatic graphs. In this paper a counterexample is given (Theorem 4). However, the special nature of the counterexample suggests that on-line 3-colorability can be tested in polynomial time, like (off-line) 2-colorability.

Conjecture 1. The problem whether $\chi^*(G)=3$ is in P.

As a consequence of the 'finite basis theorems' mentioned above, Conjecture 1 is true if restricted to triangle-free graphs, to connected graphs, or to graphs without components isomorphic to the complement of C_6. However, the complexity of deciding $\chi^*(G)=4$ probably changes.

Conjecture 2. The problem whether $\chi^*(G)=4$ is NP hard.

1. Connected on-line 4-critical graphs

Assume that A is an on-line coloring algorithm. A graph G is called *k-critical for A* if $\chi_A(G)=k$ but $\chi_A(G-x)<k$ for all $x\in V(G)$. Let FF denote the first fit algorithm which assigns the smallest legal color to the current vertex.

Notice that to show that a graph G is on-line 4-critical, one has to provide two things:

(a) *Forcing strategy of Drawer*: A sequence of the vertices of G (the next vertex given by Drawer may depend on the coloring of Painter) forcing Painter to use four colors.

(b) *Coloring strategies of Painter*: For each $x\in V(G)$ an on-line coloring algorithm $A(x)$ must be given which colors $G-x$ with at most three colors. Also there must be an on-line 4-coloring algorithm for G itself.

Without loss of generality we shall restrict the coloring strategies of Painter as follows:

— Painter uses positive integers as colors. The color of vertex x is denoted by $c(x)$.
— Painter does not make gaps in the set of used colors, i.e. she does not use color k unless she used $k-1$. (In particular, the first vertex will be colored by 1.)
— If x is the current vertex in an on-line presentation of graph G then G_x denotes the subgraph of G that was presented until this point (with x). C_x denotes the component of G_x containing x.

Proposition 1. *If G is k-critical for* FF *then $V(G)$ can be partitioned into sets C,I so that*:

(a) *I is an independent set in G and C induces a $(k-1)$-critical (for* FF*) subgraph in G*;

(b) *for all $x\in C$ there is $y\in I$ such that $xy\in E(G)$. For all $y\in I$ there is $x\in C$ such that $\Gamma(x)\cap I=\{y\}$ ($\Gamma(x)$ is the set of neighbors of x).*

Proof. If G is k-critical for FF, consider an FF coloring using k colors. Set I as the set of vertices colored by 1 and $C=V(G)\backslash I$. It is easy to check that this partition satisfies the requirements. (For the last statement use the minimality.) □

Corollary 1. *If G is k-critical for* FF *then $|V(G)|\leqslant 2^{k-1}$, consequently, there are finitely many k-critical graphs for* FF.

Then 4-critical graphs for FF are actually determined in Lemma 1. It turns out that six of them are on-line 3-colorable and the others are on-line 4-critical.

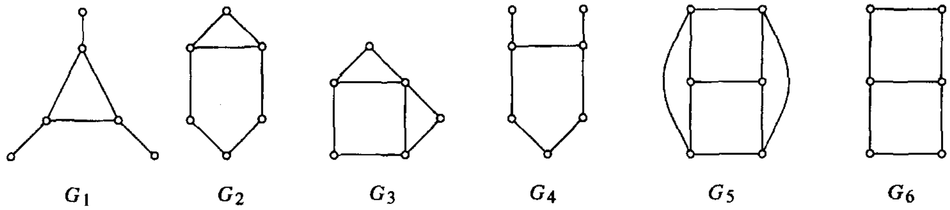

Fig. 1(a). On-line 3-chromatic graphs which are 4-critical for FF.

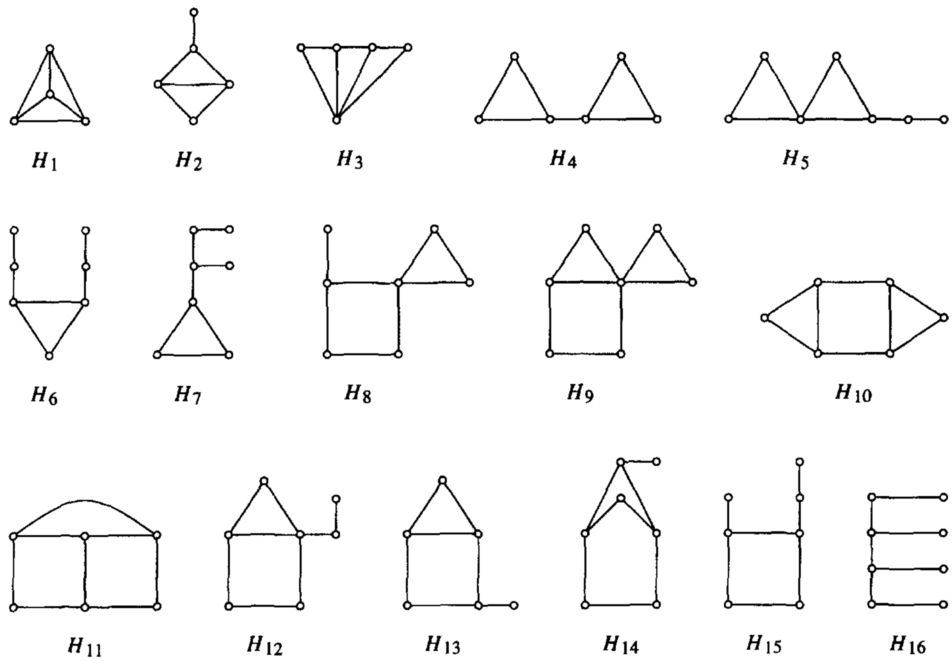

Fig. 1(b). On-line 4-critical graphs with $\chi_{FF}(H_i) = 4$.

Lemma 1. *There are* 22 4-*critical graphs for* FF. *Six of them* (G_{1-6} *on* Fig. 1(a)) *are on-line* 3-*chromatic and the others* (H_{1-16} *on* Fig. 1(b)) *are on-line* 4-*critical.* (*We will use the notation* H_{1-3} *instead of writing down* H_1, H_2, H_3.)

Proof. The finiteness of the list of 4-critical graphs for FF is clear from Corollary 1 but it takes some effort to actually determine it. One can construct the list easily by using Proposition 1 and that K_3 and P_4 are the only 3-critical graphs for FF. To see that G_i is on-line 3-colorable for $1 \leqslant i \leqslant 6$, one has to modify slightly the FF algorithm in special situations. Deviation from FF will mean that Painter tries specific coloring rules in the case of particular situations, and if a deviation rule is impossible to apply she just proceeds with FF. The on-line 3-coloring algorithms for G_i, $1 \leqslant i \leqslant 6$, are defined by the following deviation rules.

— G_1: If $C_x = \{x\}$ and G_x contains P_3 then define $c(x)$ as the color of the inner point of (any) P_3. If $C_x = \{x\}$ but G_x does not contain P_3 then define $c(x)$ as the number of components in G_x. If $G_x = P_4 + K_1$ and x is an inner point of the P_4 then color x by the color of the isolated point if possible
— G_2, G_3 and G_4:
- if $C_x = P_3$ and the midpoint of P_3 is already colored with 1 then assign $c(x)$ to make C_x colored with three different colors.
- if $C_x = P_5 = (x_1, x_2, x_3, x_4, x_5)$ and $x \neq x_3$ then try to make equal the two sets of colors $\{c(x_1), c(x_2)\}$ and $\{c(x_4), c(x_5)\}$.
- if $C_x \not\supset P_5$ but $C_x \supset P_4 = (x, x_2, x_3, x_4)$ then try to assign $c(x) = c(x_4)$, if this is not possible then try $c(x) = c(x_3)$.
- if $C_x \not\supset P_5$ but $C_x \supset P_4 = (x_1, x, x_3, x_4)$ then try to assign $c(x) = c(x_4)$.

— G_5: If $C_x \supset C_4$ then try to make the C_4 not 2-colored.
— G_6: If $G_x = 2K_2$ then define $c(x) = 3$.

It is easy to see that these algorithms color the G_is with at most 3 colors.

Next we have to show that H_i is on-line 4-critical for $1 \leqslant i \leqslant 16$. Painter's coloring strategies are trivial since $H_i - x$ is 3-colored and H_i is 4-colored by FF. It is more complicated to find Drawer's forcing strategies. Here we show only two examples for such a strategy (for the two most difficult cases), studying these it is not hard to find the strategies for the 14 simpler cases.

Forcing strategy for H_{10}: Give a vertex and then an edge component. If three colors are already used, the next vertex is joined to all previous ones. If only two colors are used, join a new vertex to two previous ones of the same color. If P_4 is colored with three colors now, a new vertex can be joined to get the fourth color. Otherwise P_4 is two-colored and the last two vertices can be given as the bottom vertices in H_{10} forcing two new colors.

Forcing strategy for H_{15}: Give two independent edges and an isolated vertex. If Painter uses 3 different colors to color them, then we can give the upper right corner of the square appropriately and force color 4 (without using the lower left corner). If they are colored by only two colors then we can suppose that the color of the isolated vertex is 1 and the edges are colored by 1 and 2. Give a new vertex connected to the isolated one and a 2-colored vertex, it will be colored by 3. Now, again, the upper right corner of the square can be given appropriately to force color 4. □

Lemma 1 implies that the description of all on-line 4-critical graphs can be achieved by finding on-line 4-critical graphs containing G_i for $1 \leqslant i \leqslant 6$. For $1 \leqslant i \leqslant 4$ and for $i = 6$ these graphs will be completely determined in a series of lemmas and they form a finite family of graphs (with 82 members, see Theorem 3). However, we could not characterize on-line 4-critical graphs containing G_5 but it is shown that there are infinitely many such graphs (Theorem 4).

Lemma 2. *If G_1 is a proper subgraph of a connected graph G then G contains one of the on-line 4-critical graphs $H_{1-3}, H_{11}, H_{18-25}$ (for the last eight see* Fig. 2).

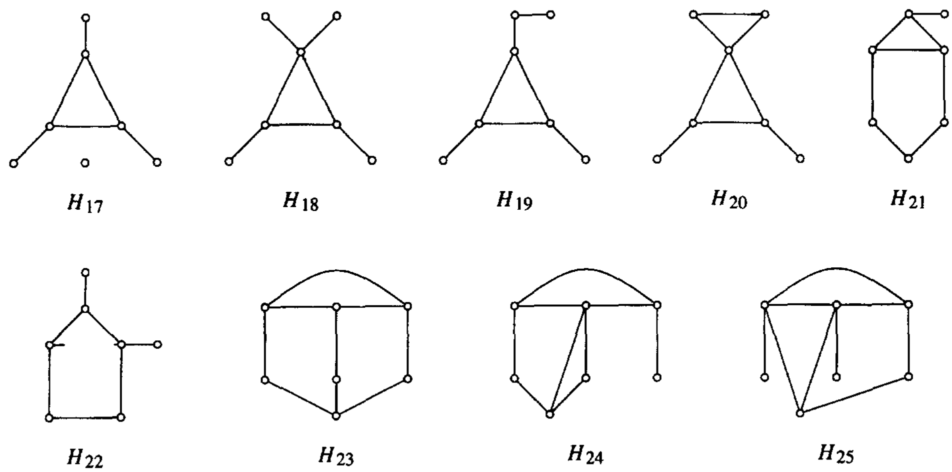

Fig. 2. On-line 4-critical graphs containing G_1.

Moreover, if G is not connected then it contains the on-line 4*-critical graph* H_{17} *(see* Fig. 2).

Proof. Assume that G_1 is a proper subgraph of G and $x \in V(G)\backslash V(G_1)$. Let T be the triangle of G_1. If x is not adjacent to any vertex of T then $x \cup G_1$ induces one of the graphs $H_{17}, H_{19}, H_{21}, H_{23}$. If it is adjacent to exactly one vertex of T then $x \cup G_1$ induces one of the graphs $H_{18}, H_{20}, H_{22}, H_{24}$ or contains H_{11}. If x is adjacent to at least two vertices of T then either $x \cup G_1$ induces H_{25} or contains one of the graphs H_1, H_2, H_3.

Next we show that H_{17-25} are on-line 4-critical.

(a) *Forcing strategy of Drawer*: Start with three independent vertices x_1, x_2, x_3.

— if $c(x_1)=c(x_2)=c(x_3)$ then we give a triangle y_1, y_2, y_3 such that y_i is adjacent to x_i.

— if $c(x_1)=c(x_2)=1$, $c(x_3)=2$ then give adjacent vertices y_1, y_2 such that y_1 is adjacent with x_1 and x_3, y_2 is adjacent with x_2 and x_3.

— if $c(x_1)=1$, $c(x_2)=2$ and $c(x_3)=3$ then the strategy depends on H_i.

- if H_i has a claw ($K_{1,3}$) (H_{18}, H_{22}, H_{23} and H_{25} have) give y adjacent to x_1, x_2, x_3.
- otherwise give y adjacent to x_3, w.l.o.g. $c(y)=1$. Give z adjacent to x_2, x_3, y. Thus color 4 is forced to z and since the graph spanned by $\{x_1, x_2, x_3, y, z\}$ is a common subgraph of $H_{17}, H_{19}, H_{20}, H_{21}$ and H_{24}, we are done.

(b) Coloring strategies of Painter: The graphs $H_i - x$ are either isomorphic to one of the graphs G_1, G_2, G_3 or do not contain 4-critical graphs for FF. Therefore either FF or one of the algorithms showed for G_i can do the work. FF colors the graphs H_i with four colors.

Lemma 3. *If* G_2 *is a proper subgraph of a connected graph* G *then* G *contains one of the on-line* 4*-critical graphs* $H_{1-4}, H_{10-11}, H_{13}, H_{21}, H_{23}, H_{26-33}$ *(for the last eight see* Fig. 3).

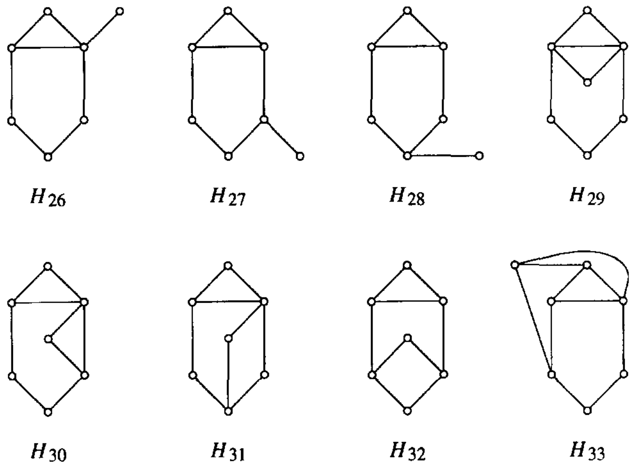

Fig. 3. Connected on-line 4-critical graphs containing G_2 (but not containing G_1).

Proof. Assume that G_2 is a proper subgraph of G. By the connectivity of G there exists $x \in V(G) \setminus V(G_2)$ adjacent to at least one vertex of G_2. If x is adjacent to exactly one vertex of G_2 then $x \cup G_2$ induces H_{21}, H_{26}, H_{27} or H_{28}. If x is adjacent to exactly two vertices of G_2 then $x \cup G_2$ either induces H_{23} or one of the graphs H_{29-32} or contains one of the graphs H_2, H_4, H_{13}. If x is adjacent to at least three vertices of G_2 then $x \cup G_2$ is either H_{33} or contains one of the graphs $H_1, H_2, H_3, H_{10}, H_{11}$.

Next it is shown that H_{26-33} are on-line 4-critical.

(a) *Forcing strategy of Drawer*: Start with two independent vertices x and y.

Case 1: $c(x)=c(y)=1$. Next nonadjacent u and v are given, both adjacent to x.

Case 1(a): $c(u)=c(v)=2$. Next w,z are given so that w is adjacent to u, y and z is adjacent to v, w, y. Clearly color 4 is forced on z.

Case 1(b): $c(u) = 2$, $c(v) = 3$. If $H_i = H_{29}$, w is given adjacent to x, y, u, v. If $H_i \in \{H_{31-33}\}$, w is given adjacent to u, v, y. Otherwise w is given adjacent to y, w.l.o.g. $c(w) = 2$. If $H_i \in \{H_{26}, H_{27}, H_{30}\}$ then z is given adjacent to x, v, w. If $H_i = H_{28}, z$ is given adjacent to v, y, w. In all subcases 4 colors are forced.

Case 2: $c(x)=1$, $c(y)=2$. Next u is given adjacent to x. If $c(u)=3$ then z is given adjacent to x, y, u and we are done. If $c(u)=2$ then v is given adjacent to u. If $c(v)=1$ then Case 1(a), if $c(v)=3$ then Case 1(b) can be applied by reversing the role of the colors 1 and 2.

(b) *Painter's coloring on* $H_i - x$: These graphs are either isomorphic to G_2 or do not contain 4-critical graphs for FF. Thus FF or the algorithm for G_2 can do the work. Moreover, FF colors the graphs H_i with four colors. □

The proofs of the next three lemmas are similar to the two previous ones (they are based on case analysis) so they are omitted.

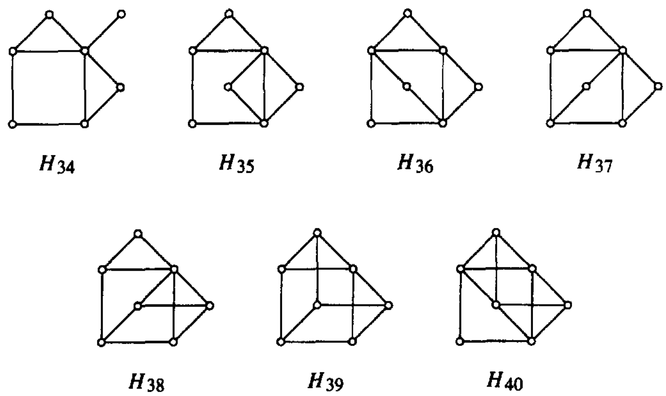

Fig. 4(a). Connected on-line 4-critical graphs containing G_3 (but not containing G_1).

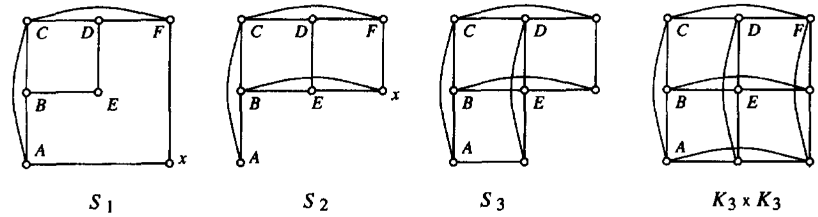

Fig. 4(b). Components containing G_3 in an on-line 3-colorable graph.

Lemma 4. *If G_3 is a proper subgraph of a connected graph G then either G contains one of the on-line 4-critical graphs $H_{1-4}, H_{10-11}, H_{13}, H_{24}, H_{34-40}$ (for the last seven see* Fig. 4(a)) *or G is one of the on-line 3-colorable graphs $S_1, S_2, S_3, K_3 \times K_3$ (see* Fig. 4(b)).

Lemma 5. *If G_4 is a proper subgraph of a connected graph G then either G contains one of the on-line 4-critical graphs $H_2, H_3, H_7, H_{11}, H_{13-16}, H_{26}, H_{27}, H_{41-46}$ (for the last six see* Fig. 5(a)) *or G is one of the on-line 3-chromatic graphs S_4, S_5, S_6, S_7 or the Petersen graph P (see* Fig. 5(b)). *In fact, S_4 denotes an infinite family of graphs, the number of P_4s and pendant edges are arbitrary. However, vertices E and D are of degree at least three.)*

Lemma 6. *If G_5 is a proper subgraph of a connected graph G then either G contains one of the on-line 4-critical graphs $H_{1-3}, H_{11}, H_{13}, H_{15}, H_{24}, H_{35}, H_{47}$ (see* Fig. 6) *or G is one of the on-line 3-chromatic graphs $S_2, S_3, K_3 \times K_3$ (see* Fig. 4(b)).

For the next section we need the following corollary of Lemmas 2–6.

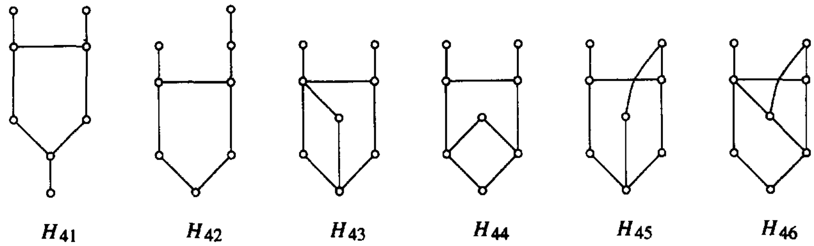

Fig. 5(a). Connected on-line 4-critical graphs containing G_4.

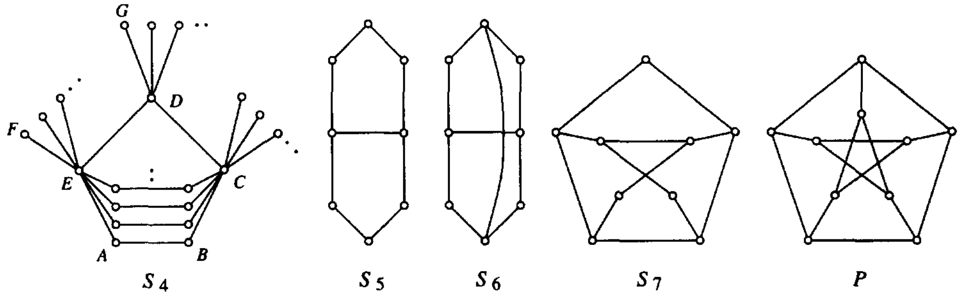

Fig. 5(b). Components containing G_4 in an on-line 3-colorable graph.

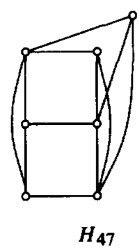

Fig. 6. The only connected 4-critical graph containing G_5.

Corollary 2. *If a connected graph G is on-line 3-chromatic and contains a subgraph from G_1 to G_5 then $\chi_{FF}(G) = 4$. Moreover, if G contains C_6 then G is a subgraph of one of the following graphs*: S_4, P (*on* Fig. 5(b)), $K_3 \times K_3$ (*on* Fig. 4(b)).

Proof. Lemmas 2–6 imply that G is one of the graphs $G_1, G_2, S_1, S_2, S_3, K_3 \times K_3, S_4, S_5, S_6, S_7$ or P. Note that S_1, S_2, S_3 are subgraphs of $K_3 \times K_3$ and S_6 and S_7 are both subgraphs of P, so G is a subgraph of G_1, G_2, S_4, S_5, P or $K_3 \times K_3$. Since G_1, G_2 and S_5 do not contain C_6, the second part is obvious. For the first part it is enough to prove that FF colors these 6 maximal graphs with at most 4 colors. For this we have to prove that if FF gets the graph in any order the new vertex can never have four differently colored neighbors. This is trivially true for G_1, G_2, S_5 and P (because the maximum degree is three). For S_4: when getting one of E, D or C all the existing pendant neighbors are colored by 1 and all the possible existing P_4 neighbors are colored by 1 or 2. For $K_3 \times K_3$: as there are no two vertices of the same color in any row or column, by the FF rule there must be a 1-colored vertex in every row and in every column, so all the other vertices have two 1-colored neighbors. □

In Lemma 7 the remaining case of G_6 is discussed. The proof is much more difficult than the brute force methods of the previous lemmas. The reason is that a connected component of an on-line 3-chromatic graph containing only G_6 from the set $\{G_i\colon 1 \leqslant i \leqslant 6\}$ can be very complicated. The structure of these graphs have been described in [1] for the triangle-free case. For triangle-free graphs containing G_6 the following results have been proved in [1].

Theorem A (Gyárfás et al. [1, Theorem 3]). *The following statements are equivalent for any connected triangle-free graph G containing G_6:*

(1) *G is on-line 3-chromatic;*

(2) *G is $(2K_2+K_1)$-free;*

(3) *G does not contain subgraphs isomorphic to $H_{14}, H_{15}, H_{48-51}$.*

Theorem B (Gyárfás et al. [1, Theorem 4]). *Let G be a triangle-free graph containing G_6. Then G has on-line chromatic number three iff G has no subgraph isomorphic to any of H_{14}, H_{15} and H_{48-57}.*

Lemma 7. (a) *If G_6 is a proper subgraph of a connected graph G then either G contains one of the on-line 4-critical graphs $H_{2-3}, H_8, H_{11}, H_{13-15}, H_{48-51}, H_{58}$ (for the last five see* Fig. 7) *or G is on-line 3-colorable and $(2K_2+K_1)$-free and triangle-free.*

(b) *If a disconnected graph G contains G_6 and on-line 3-chromatic then it has at most two non-trivial components. The component containing G_6 is $(2K_2+K_1)$-free and triangle-free, the second component is either a triangle or a five-cycle or a complete bipartite graph or a complete bipartite graph minus one edge. Moreover, if G is disconnected, contains G_6 and the components of G are on-line 3-colorable then either G is on-line 3-colorable or it contains one of the on-line 4-critical graphs $H_{52-57}, H_{59,60}$ (on* Fig. 7).

Proof. First we should prove that the graphs on Fig. 7 are on-line 4-critical but this part is left to the reader.

Consider part (a). Suppose that the connected graph G containing a copy of the G_6 also contains a triangle. If a copy of G_6 and a copy of K_3 intersect each other in two vertices then G must contain one of $(H_2, H_3, H_{11}, H_{13}, H_{58})$. If there are no such copies but there are copies of G_6 and K_3 intersecting in one vertex then G contains one of (H_2, H_8, H_{48}). If all copies of G_6 are disjoint from all triangles then either there is a triangle fully connected to a copy of G_6 (all three vertices of triangle are adjacent to at least one vertex of G_6) or there is a vertex which is of distance two from G_6. In the first case H_{48} and in the second case one of (H_{15}, H_{48}, H_{50}) must occur as a subgraph of G.

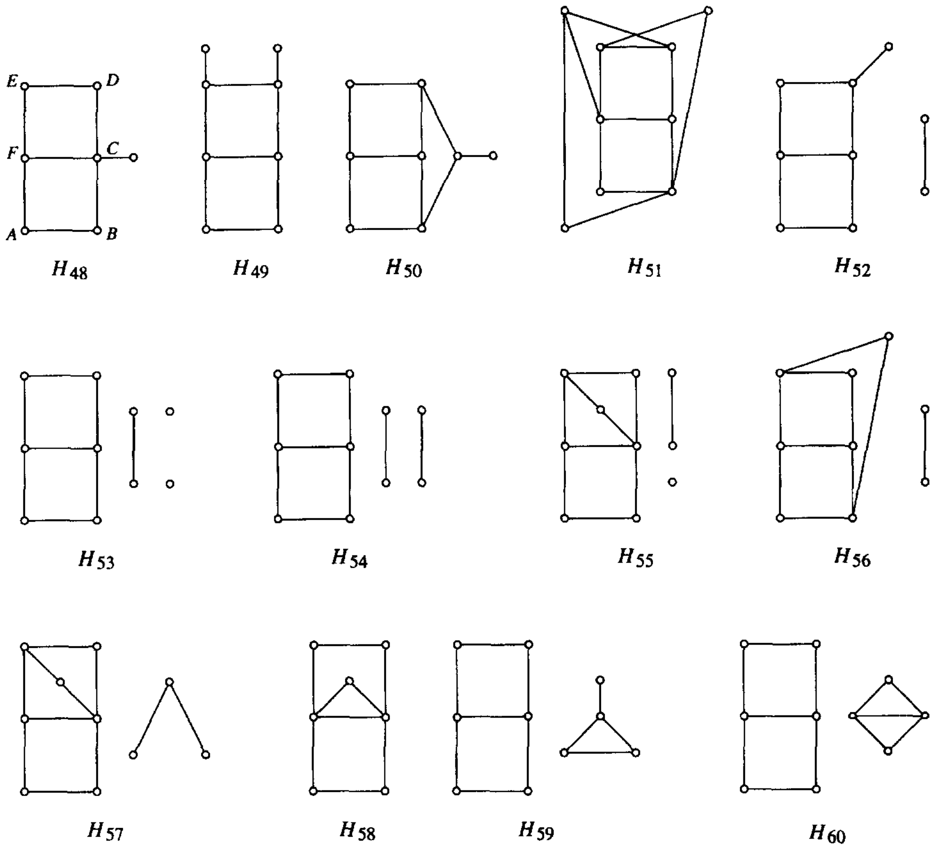

Fig. 7. On-line 4-critical graphs containing G_6.

If G does not contain any triangle then we apply Theorem A. This completes the proof for the connected case.

For proving part (b) first observe that because H_{54} is forbidden in an on-line 3-chromatic graph there cannot be more than two non-trivial components. The statement for the component containing G_6 follows from the previous part of the proof. If the second component contains a triangle then it must be the triangle itself because H_1, H_{59} and H_{60} cannot be a subgraph of an on-line 3-chromatic graph. If it does not then, because the graphs H_{53} and H_{54} are on-line 4-chromatic, the second component must be either C_5 or a complete bipartite graph with at most one edge deleted.

For proving the last statement of part (b) first consider the case when G does not contain any triangle. In this case the statement is a consequence of Theorem B as H_{14}, H_{15} and H_{48-51} are connected graphs but not on-line 3-colorable ones. Next consider the case when G contains a triangle. As the components are on-line 3-colorable a component containing a triangle cannot contain G_6 and H_1; so either it is a triangle itself or G contains H_{59} or H_{60}. If there are two components containing a triangle then G contains H_{54}. So it remains to show that if G has a triangle component, all components of it are on-line 3-colorable but G does not contain any of the graphs H_{52-57}, H_{59-60}, then G is on-line 3-colorable. Let x be a vertex of the triangle. By the previous case there is an on-line 3-coloring algorithm for $G - x$. Observe that essentially the same algorithm will be good for G (possibly adding the rule that if the current vertex form a triangle then color it by FF and later 'forget' about this vertex). □

Theorem 1. *A connected graph is on-line* 3*-colorable if and only if it does not contain any subgraph isomorphic to* H_{1-16}, H_{18-51}, H_{58}. *These graphs are on-line* 4*-critical and* FF *colors them by* 4 *colors. Consequently, there are* 51 *connected on-line* 4*-critical graphs and on-line* 3*-colorability can be checked in polynomial time for connected graphs.*

Proof. The 'only if' part is obvious since the list contains on-line 4-critical graphs. It is also easy to check that FF colors all of them by 4 colors. The consequences are also immediate. To prove the 'if' part, assume that G is a connected graph without subgraphs of the list of the theorem. If $\chi_{FF}(G) = 3$, we have nothing to prove. Otherwise G must contain a 4-critical graph H for FF. Lemma 1 implies that H is either G_i $(1 \leqslant i \leqslant 6)$ or H_i $(1 \leqslant i \leqslant 16)$. The latter case is excluded by the assumption of the theorem. Therefore H is isomorphic to G_i for some i, $(1 \leqslant i \leqslant 6)$. Applying Lemma $i+1$ and the statement from Lemma 1 which says that the graphs G_1–G_6 themselves are on-line 3-colorable, we get that G is 3-colorable (in case of $i \leqslant 5$ G is a subgraph of one of the following graphs: $G_1, G_2, S_4, S_5, P, K_3 \times K_3$).

2. Coloring on-line 3-chromatic graphs

It was shown in [5] that it is not possible to have *one* on-line coloring algorithm which can color all on-line 3-chromatic graphs with 3 colors. In this section we will show that (a bit surprisingly) a simple modification of FF can color all these graphs with 4 colors. Why can these types of theorems be useful or interesting? We know from several examples that an on-line coloring algorithm cannot be good compared to the off-line one (i.e. the chromatic number of the graph); for example, there are trees with arbitrary large on-line chromatic number. The case is even worse and Szegedy [9] showed that for any on-line coloring algorithm a graph can be constructed which has n vertices, $O(\log n)$ chromatic number but the algorithm will use $\Omega(n/\log n)$ colors on it. So because we cannot find good on-line coloring algorithms (compared to the off-line ones), one possibility is to find good algorithms compared with all the other on-line algorithms. Our result states in other words that if for a graph *there exists* an on-line algorithm which colors it by 3 colors, then our algorithm *can color that graph by* 4 *colors*.

The algorithm is the following:

Algorithm FF(C_6). Use FF with the following exception rules:
— if the new vertex would be colored by 3 but can be colored by 4
— and if there is an (induced) C_6 containing the new vertex in which the opposite vertex is colored by 3
then color the new vertex by 4.

Theorem 2. FF(C_6) *colors each on-line* 3*-chromatic graph with at most* 4 *colors.*

Proof. First we observe that it is enough to prove that the algorithm will color all *connected* on-line 3-chromatic graphs with at most 4 colors. (In this case it will color properly all of the components).

If FF colors a connected graph with 3 colors then FF(C_6) will color it with at most 4 colors. So by Lemma 1 it is enough to consider the on-line 3 chromatic connected graphs which contain G_1, G_2, G_3, G_4, G_5 or G_6.

If it contains one of the G_1 to G_5 then by Corollary 2 FF would color it with 4 colors. If FF(C_6) did not use the exception rule then it behaves the same as FF. If it used the exception rule then the graph must contain a C_6 so it is a subgraph of S_4 or P or $K_{3,3}$ (Corollary 2 again). It is not too hard to check that FF(C_6) colors these three graphs with 4 colors. For P this is obvious since the degrees are at most three. For $K_3 \times K_3$ the proof given in the proof of Corollary 2 works without any modification. For the case of S_4 first observe that the pendant edges from one vertex can be replaced by one pendant edge since the pendant neighbors of one vertex will be colored by the same color. If FF(C_6) colored a vertex by 5 then this vertex must have degree at most four and must be a later coming neighbor of a 4-colored vertex which was colored using the exception rule. It is easy to see that this is impossible.

The only remaining case is that the connected graph contains G_6. From Lemma 7 we know that the graph is triangle-free and it is $(2K_2+K_1)$-free. From now on let G denote the connected on-line 3-colorable $(2K_2 + K_1)$- and triangle-free graph containing G_6. Our purpose is to show that FF(C_6) will color G with at most 4 colors. □

Proposition 2. *At the end of the coloring by* FF(C_6) *the colored graph G cannot contain a C_6 which has two opposite vertices colored by* 3.

Proof. If G does not contain C_6 then the statement is obvious. So we restrict ourselves to the case when G contains C_6. First we observe that since the graph G is $(2K_2+K_1)$- and triangle-free, G has the strict structure shown on Fig. 8. All of the vertices in set A must have two or three of the dashed edges starting from A and the same is true for set B. Additionally, there can be an optional bipartite graph between A and B (dotted lines) but no edges inside A or B. Therefore G is bipartite.

Suppose that the proposition is not true, let $\{u, v\}$ be the first pair (in the order we got G) of 3-colored vertices that will be opposite vertices of a C_6. As our algorithm

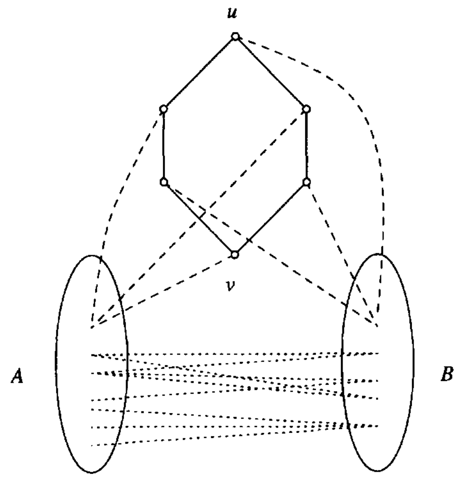

Fig. 8. The structure of G if it contains C_6.

colored them by 3, both of them were an inner point of a 1–2–3–1 colored P_4 when the color 3 was assigned to them. Because of the bipartiteness we know that u and v are in different partite classes of the bipartite graph so there are only 3 possibilities how the two P_4s are embedded in G: either they are disjoint or they have 1 endpoint in common or their both endpoints are common. First we claim that at the moment when we were given the second of $\{u, v\}$ (let us say v) they were opposite vertices of a C_6. If the two P_4s have both of their endpoints in common then the statement is obvious. In the other two cases we use that G does not contain $(2K_2+K_1)$, so there must be additional edge(s) between the P_4s and we get again a C_6.

Now using the structure described in the first paragraph of the proof we can assume that the C_6 of which u and v were the opposite vertices is the C_6 drawn separately on Fig. 8. We colored v by 3 so we did not use the exception rule therefore v had a neighbor w colored by 4 when it arrived. Look at the picture on Fig. 8. w is either on the C_6 or in A but in both cases before getting w (and so before v) there was a 3-colored vertex v' in B (since the cause of coloring w by 4 is a 3-colored vertex by 1 or 3 distance from w) which contradicts the choice of $\{u, v\}$, because $\{u, v'\}$ are 3-colored and opposite vertices of a C_6 (u and v' are colored by 3 so there is no edge between them therefore the other two dashed edges must be presented) and came earlier. This is the end of proof of the proposition.

Now we are ready to finish the proof of the theorem. We consider two cases:

Case 1: *The algorithm did not use the exception rule.* In this case if the algorithm failed then the graph must contain a 5-critical graph for FF. It is easy to determine all of the 5-critical graphs for FF which are on-line 3-colorable, contain G_6 and are $(2K_2+K_1)$- and triangle-free. Just one such graph exists, which can be seen on Fig. 9. However, it has a C_6 which has two opposite vertices colored by 3, so by Proposition 2 it could not appear.

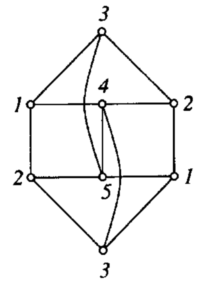

Fig. 9. The only 5-critical graph for FF which is on-line 3-colorable, contains G_6 and is $(2K_2 + K_1)$- and triangle-free.

Case 2: *The algorithm used the exception rule.* In this case we have the structure described in Proposition 2. Now we can suppose that u is colored by 3 and v is colored by 4. By Proposition 2 there cannot be 3-colored vertex in set B. But in this case there cannot be 4-colored vertex in set A because it has neither 3-colored neighbors nor a 3-colored vertex with distance 3. So there cannot be a 5-colored vertex either in A or in B or on C_6; consequently, $\mathrm{FF}(C_6)$ used at most 4 colors. □

3. Disconnected on-line 4-critical graphs

Notice that in Lemma 2 and in Lemma 8 all disconnected on-line 4-critical graphs containing G_1 or G_6 had been determined ($H_{17}, H_{52-57}, H_{59}, H_{60}$). Our next objective is to determine all disconnected on-line 4-critical graphs containing G_2 or G_3 or G_4. We shall see that there are finitely many such graphs. (However, in Section 4 we shall give infinitely many on-line 4-critical graphs containing G_5.)

Let A denote the family of graphs not containing induced subgraphs isomorphic to G_1 or to G_6 but containing at least one induced subgraph isomorphic to either G_2 or to G_3 or to G_4. A component of a graph in A is called *major component* if it has an induced subgraph isomorphic to one of the graphs G_2, G_3, G_4. Other components are called *minor components.* Notice that Lemmas 3–5 and the definition of A immediately gives

Lemma 8. *Assume that $G \in A$ and G does not contain induced subgraphs H_{26-47}. Then all major components of G are isomorphic to one of the following graphs: $G_2, G_3, S_1, S_2, S_3, K_3 \times K_3, G_4, S_4, S_5, S_6, S_7, P$.*

The following slight modification of the FF algorithm is called VRFF (V-restricted first fit): Follow FF except if the current vertex x is creating a P_3 component whose center is colored by 1. In this case color x with color 3 (instead of color 2 assigned by FF).

Lemma 9. *The graphs N_i, $1 \leqslant i \leqslant 6$ (see* Fig. 10) *are 4-critical for* VRFF.

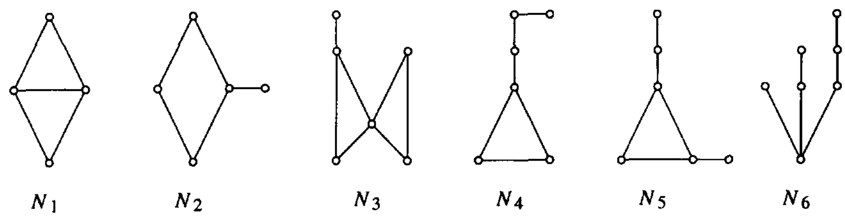

Fig. 10. Some 4-critical graphs for VRFF algorithm.

Proof. Immediate by inspection.

The next lemma gives 18 new on-line 4-critical graphs in A.

Lemma 10. *The* 18 *graphs* $M_{i,j}$ *with major component* G_i, $2 \leqslant i \leqslant 4$, *and with minor component* N_j, $1 \leqslant j \leqslant 6$, *are on-line* 4-*critical.*

Proof. Intuitively, the forcing strategy of Drawer on $M_{i,j}$ is force four colors on the major component if FF is used by Painter and force four colors on the minor component if VRFF is used by Painter. The pigeonhole principle is used to find remedy against mixed strategies. In fact, the graphs $M_{i,j}$ are large enough to start with isolated vertices and edges to force a choice between FF and VRFF according to the demands of G_i and N_j. The technique is illustrated with two examples.

Example 1 ($M_{2,1}$). Drawer gives isolated vertices until two of them are colored with the same color, say color 1. If this happens immediately then two new vertices are given, both adjacent to the first vertex. Now a fourth color can be forced on G_2 if the two new vertices colored with the same color and on N_1 otherwise. If two vertices are colored with the same color only after three or four isolated vertices are given then G_2 is used for forcing and N_1 can keep the unwanted vertex.

Example 2 ($M_{4,6}$). Summary of the forcing strategy: Drawer starts with four isolated vertices. If three colors are used by Painter, the fourth can be forced on a claw of G_4. If two colors used in 2–2 distribution, Drawer uses G_4 again to force four colors. If three vertices have the same color (say color 1) then two new vertices (x, y) are given, both adjacent to the same vertex of color 1. If $c(x) = c(y)$ then G_4 is used for forcing four colors. Otherwise a new vertex is given adjacent to a different vertex of color 1 and four colors are forced on N_6.

To see that $M_{i,j} - x$ is on-line 3-colorable for all $x \in V(M_{i,j})$, note that for $x \in V(G_i)$ the FF algorithm works and for $x \in V(N_j)$ the VRFF algorithm works. □

The next lemma describes the possibilities for minor components of graphs in A if $M_{i,j}$ and some other on-line 4-critical subgraphs are forbidden.

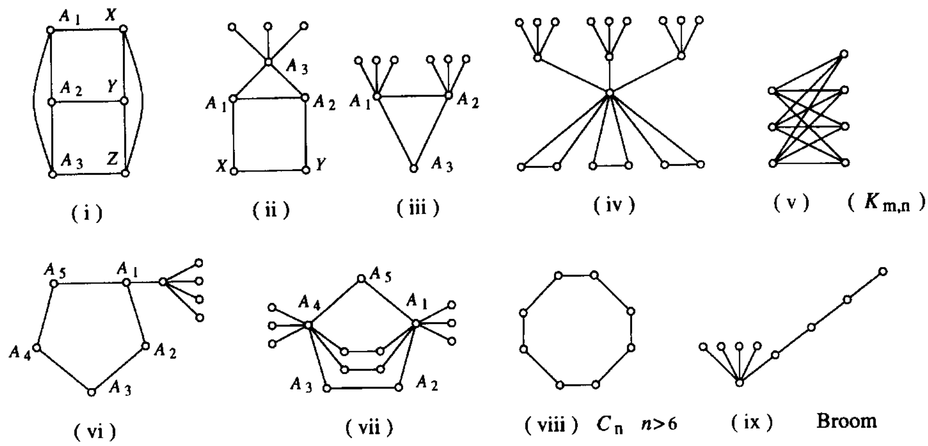

Fig. 11. Minor components.

Lemma 11. *Assume that* $G \in A$ *and* G *does not contain any subgraph from the list* $\{H_1, H_4, H_{10}, H_{47}\} \cup \{M_{i,j}\colon 1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 6\}$. *Then minor components of* G *are isomorphic to an induced subgraph of some graph on* Fig. 11. (*For graphs on* Fig. 11 *the number of pendant vertices, the number of triangles* (*in* (iv)) *and the number of 'inner'* P_4*s between* A_1 *and* A_4 (*in* (vi)) *can be any positive number, in* (ix) *the long path can be arbitrarily long and case* (v) *and* (viii) *mean as written.*)

Proof. Let C be a minor component of G. Notice that in addition to $(H_1, H_4, H_{10}, H_{47})$, C does not contain N_j for $1 \leqslant j \leqslant 6$. The definition of the minor component (and the definition of A) implies that C does not contain G_1, G_2, G_3, G_4 and G_6. These conditions determine C, a systematic way is to subdivide the cases according to $g(C)$, the girth of C.

Case 1: $g(C) = 3$. Let T be a triangle of C with vertices A_1, A_2, A_3. Since H_1 and N_1 are forbidden in C, the sets $U_i = \{x \in V(C) \backslash V(T)\colon\ x \text{ is a neighbor of } A_i\}$ are pairwise disjoint.

Case 1(a): For some $i \neq j$, $x \in U_i$, $y \in U_j$ and $xy \in E(G)$. Now $|U_i| = |U_j| = 1$ otherwise G_3 or N_1 or N_2 is a subgraph of C. By symmetry, we may assume $x \in U_1$, $y \in U_2$, $|U_1| = |U_2| = 1$. If $z \in U_3$ and $|\Gamma(z) \cap \{x, y\}| = 1$ then N_2 is a subgraph of C. If $|\Gamma(z) \cap \{x, y\}| = 2$, i.e. both zx and zy are edges of G then $T \cup \{x, y, z\}$ induces G_5 in C. Therefore either $C = G_5$ which is case (i) in Fig. 11 or Lemma 6 can be applied to C which says that C contains some graphs isomorphic to $H_{1-3}, H_{11}, H_{13}, H_{15}, H_{24}, H_{35}, H_{47}$ or G is S_2, S_3 or $K_3 \times K_3$. But all of this subgraphs contain either H_1 or N_1 or N_2 or G_3 so the only possibility is that $C = G_5$. Therefore for the remaining case for all $z \in U_3$, $|\Gamma(z) \cap \{x, y\}| = 0$. Moreover, for $z_1, z_2 \in U_3$, $z_1 z_2 \notin E(G)$; otherwise $T \cup \{z_1, z_2, x\}$ induces N_3. This means that $Z = V(T) \cup U_1 \cup U_2 \cup U_3$ induces a subgraph like (ii) on Fig. 11. We are going to show that $Z = V(C)$. Indeed, if there are other vertices in C then the connectivity of C allows to choose $t \in V(C) \setminus V(Z)$ such that $\Gamma(t) \cap Z = \Gamma(t) \cap (\{x,y\} \cup U_3) \neq \emptyset$. But if t has some neighbors in $\{x,y\}$ then N_2 or H_{10} is induced in C. On the other hand, if t has neighbors only in U_3 then N_5 is induced in C. This proves $Z = V(C)$ and shows that Case 1(a) implies that C is either (i) or (ii) (or a subgraph of them).

Case 1(b): $xy \notin E(G)$ for $x \in U_i$, $y \in U_j$ if $i \neq j$. Notice that $G_1 \nsubseteq C$ implies $U_i = \emptyset$ for some i, say $U_3 = \emptyset$. If both U_1 and U_2 are non-empty then they span independent sets from the condition $N_3 \nsubseteq C$. We are going to show that $Z = V(T) \cup U_1 \cup U_2$ contains all vertices of C, i.e. C is the graph shown as (iii) on Fig. 11. If not, the connectivity of C implies that there exist $t \in V(C) \setminus Z$ such that $\emptyset \neq \Gamma(t) \cap Z \subset U_1 \cup U_2$. Now if t has neighbors both in U_1 and U_2 then C contains G_2, contradiction. If t has neighbors just one of the sets U_1 and U_2 then C contains N_5, contradiction again. Therefore C is the graph of type (iii). Finally, if both U_2 and U_3 are empty then U_1 induces a subgraph in C whose components are edges and single vertices because H_1 and N_1 are forbidden. Let W_1 be the union of the edge components and let W_2 be the union of the single vertices in U_1. Let t be a vertex of C not in T nor in U_1. As N_1, N_2 and N_3 are forbidden, t does not have neighbors in W_1. If t has two neighbors in W_2 then N_2 would be a subgraph of C. If t does not have any neighbors in W_2 then by connectivity a copy of N_4 would appear. So all the vertices of C not in T nor in U_1 have exactly one neighbor in W_2. Finally, as $H_4 \not\subset C$ and $N_4 \not\subset C$, there are no edges between any two vertices of this type. Therefore C has a structure as (iv) on Fig. 11. We conclude that Case 1(b) implies C is either (iii) or (iv) or a subgraph of them.

Case 2: $g(C) = 4$. It is immediate that $N_2 \not\subset C$ implies that C is a complete bipartite graph, i.e. graph (v) on Fig. 11.

Case 3: $g(C) = 5$. Let T be a C_5 in C with vertices $A_1A_2A_3A_4A_5$. Let U be the set of vertices in $V(C) \setminus V(T)$ adjacent to some vertex of T. Since $g(C) = 5$, all vertices of U are adjacent to exactly one vertex of T. Thus $U = \bigcup_{i=1}^{5} U_i$, where $U_i = \{x \in V(C) \setminus V(T)\colon\ \Gamma(x) \cap T = \{A_i\}\}$. The sets U_i are independent and if $i - j \equiv 1 \pmod 5$ then U_i or U_j is empty otherwise $G_4 \subset C$. This means that $U_i \neq \emptyset$ for at most two non-consecutive values of i.

Case 3(a): $U_2 = U_3 = U_4 = U_5 = \emptyset$. If $|U_1| \geqslant 2$ then $V(C) = T \cup U_1$; otherwise $N_6 \subset C$. Therefore C is a subgraph of (vii). If $|U_1| = 1$, say $U_1 = \{x\}$ then all vertices of $V(C) \setminus (T \cup U_1)$ are adjacent to x and so C is as shown on (vi) (again from the condition $N_6 \not\subset C$). If U_1 is also empty then $C = T$ and it is covered by both (vi) and (vii).

Case 3(b). $U_1 \neq \emptyset$, $U_4 \neq \emptyset$ and so $U_2 = U_3 = U_5 = \emptyset$. Now from the condition $N_6 \not\subset C$ it follows that $V(C) = T \cup U_1 \cup U_4$. The condition $g(C) = 5$ implies that between U_1 and U_4 there are independent edges only. Thus C is (vii) shown on Fig. 11.

Case 4. $g(C) = 6$. As N_6 is forbidden and there are no shorter cycles, C is obviously a six-cycle with possible pendant edges at two opposite vertices, so it is a subgraph of (vii).

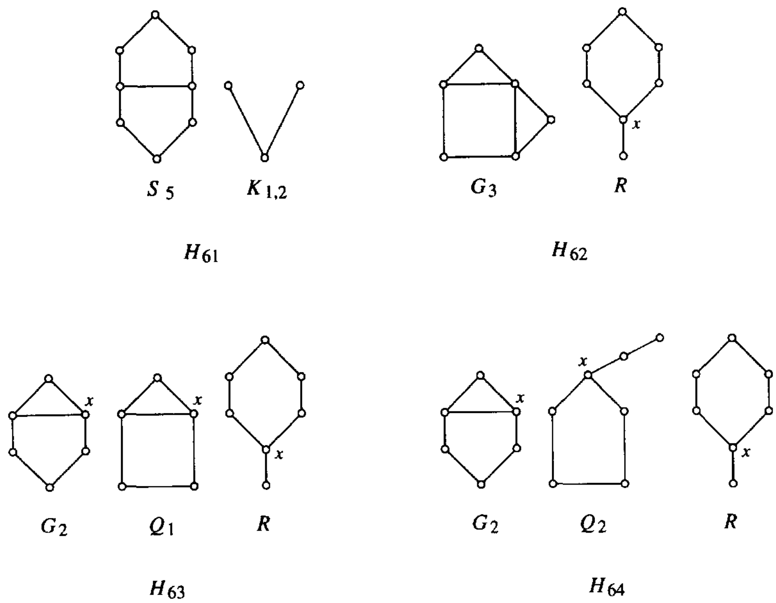

Fig. 12. Special 4-critical graphs.

Case 5. $7 \leqslant g(C) < \infty$. Now C must be a cycle as (viii) on Fig. 11 because N_6 is forbidden.

Case 6. $g(C) = \infty$. Now C is a tree without N_6 subtree, so C is either a tree of radius 2 and so C is a subgraph of (iv) or C is a 'broom' (or a subgraph of it, e.g. a path) shown on (ix). □

Lemmas 8 and 11 show that a graph in A either contains an on-line 4-critical graph from a 'small' list or it is completely described (for giving the possibilities for its major and minor components). The next lemma says that among these completely described graphs there are only four on-line 4-critical graphs. The proof is rather straightforward but lengthy so it is omitted.

Lemma 12. *Assume that G is a graph whose components are either $G_2, G_3, S_1, S_2, S_3, K_3 \times K_3, G_4, S_4, S_5, S_6, S_7, P$ or a subgraph of one of the graphs on* Fig. 11. *Then G is either on-line* 3*-colorable or contains one of the on-line* 4*-critical graphs H_{61-64}* (*see* Fig. 12).

The results proved so far are summarized in the following theorem. Let L denote the set of the 82 graphs H_{1-64} plus $M_{i,j}$ $(2 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 6)$.

Theorem 3. *For any graph G*
— *either G is on-line* 3*-colorable*
— *or G contains a subgraph from L*
— *or for any component C of G either $C = G_5$ or $\chi_{FF}(C) \leqslant 3$.*

Proof. Assume that $\chi^*(G) \geqslant 4$. Then G contains an H which is 4-critical for FF. If $H = H_i$ for $1 \leqslant i \leqslant 16$ then $G \supset H \in L$. If $H = G_1$ or $H = G_6$ then Lemma 2 or Lemma 7 implies that G must contain a subgraph from L. If $H = G_2$ or $H = G_3$ or $H = G_4$ then $G \in A$ and Lemmas 8, 11, 12 imply that G contains a subgraph from L.

Thus we assume $H = G_5$ and every component of G is G_i-free for $i = 1, 2, 3, 4, 6$. It is not possible that a component of G strictly contains G_5 because by Lemma 6 it would be isomorphic to S_2, S_3 or $K_3 \times K_3$ but these graphs contain G_3. Consequently, for every component C of G either $C = G_5$ or $\chi_{\text{FF}}(C) \leqslant 3$. □

Corollary 3. *If G is on-line 4-critical then either $G \in L$ or for any component C of G $C = G_5$ or $\chi_{\text{FF}}(C) \leqslant 3$ (both types must be represented in G).*

Notice that although 4-critical graphs are far from being characterized completely, Corollary 3 gives some interesting properties. For example, it is easy to check that $\chi_{\text{FF}}(G) = 4$ for all $G \in L$. This leads to

Corollary 4. FF *colors on-line 4-critical graphs by 4 colors.*

The following result is seemingly trivial but we do not see any simple proof of it.

Corollary 5. *If G is on-line 5-critical then $\chi^*(G - x) = 4$ for some $x \in V(G)$.*

Proof. Apply Theorem 3 for G. Since G is not on-line 3-colorable, G must have either a subgraph from L or for any component C of G, $\chi_{\text{FF}}(C) \leqslant 4$. The latter can not happen since in that case $\chi^*(G) \leqslant \chi_{\text{FF}}(G) \leqslant 4$ — contradiction. Therefore there is a subgraph H of G such that $H \in L$. Now, using that G is critical, for any vertex $x \in V(G) \backslash V(H)$ $\chi^*(G - x) = 4$. □

It is easy to see that the same argument can be used to prove that any k-chromatic ($k \geqslant 4$) graph must contain a 4-critical subgraph. So we get the following Corollary which shows that the set of 4-critical graphs 'describes' the set of on-line 3 colorable graphs.

Corollary 6. *A graph G is on-line 3-colorable iff it does not contain a 4-critical graph.*

Conjecture 3. For every k and for every k-critical graph G there exists $x \in V(G)$ such that $\chi^*(G - x) = k - 1$.

Or in an equivalent form:

Conjecture 3′. For every graph G there exists a vertex x such that $\chi^*(G - x) \geqslant \chi^*(G) - 1$.

4. An infinite family of on-line 4-critical graphs

Theorem 3 shows that on-line 4-critical graphs different from the 82 described before are rather special graphs: their components are either G_5 or 3-colorable by FF. Throughout this section the G_5 components are called major components, the others are called minor components. A C_4R algorithm is an on-line coloring algorithm with the following property: if the current vertex creates a component isomorphic to C_4 (the four cycle) then this C_4 component must be colored with three different colors. A special case of a C_4R algorithm is the C_4RFF algorithm where FF is applied whenever possible. The next lemma gives some new on-line 4-critical graphs.

Lemma 13. *The graphs* $G_5 \cup Z_i$ *are on-line* 4*-critical for* $1 \leqslant i \leqslant 7$ (*see* Fig. 13).

Proof (*outline*). The forcing strategy of Drawer is to give sufficiently many isolated vertices and then build a C_4. If the C_4 is colored with two colors, four colors can be forced on the major component, otherwise on the minor component. On the other hand, if a vertex x is removed from $G_5 \cup Z_i$ then in case of $x \in V(G_5)$ FF colors with three colors; in case of $x \in V(Z_i)$, C_4RFF colors with three colors. □

In the spirit of Lemma 13 we shall define infinitely many on-line 4 critical graphs. Set $G(0) = G_5$ and let $G(1)$ be defined as shown on Fig. 14. Moreover, for $i = 2, 3, \ldots,$ let $G(i)$ be the graph shown on Fig. 15 which is called the i-kite. Finally, the graph $G^*(i)$ is defined by adding a pendant edge to the i-kite as shown on Fig. 16.

Theorem 4. *The graph*

$$K(n) = \bigcup_{i=0}^{2n-1} G(i) \cup G^*(2n)$$

is on-line 4*-critical for each* $n \geqslant 1$.

Proof. The forcing strategy of Drawer on $K(n)$ is the following. Start with five isolated vertices, Painter must color three of them (say x, y, t) with the same color, say color 1 (otherwise four colors can be forced on a claw). If there are four among them with color 1 let us denote the fourth one by z and give the next vertex u, adjacent to z and let $c(u) = 2$. If the other two vertices are colored by a different color, say 2, then denote one of them by u and give a neighbor z of it (just for uniformity). Continue with vertices v and w, both adjacent to x and y. If Painter colors so that $c(v) = c(w)$ then the fourth color can be forced on the component $G(0) = G_5$. So we may assume that $c(v) = 2$, $c(w) = 3$ (see Fig. 17(a)).

Vertex p_1 is given next which is adjacent to v. If $c(p_1) = 1$ then a new vertex adjacent to v, p_1, w forces the fourth color on the $G(1)$ component. Therefore $c(p_1) = 3$. The next vertex is q_1, adjacent to w. If $c(q_1) = 2$ then a new vertex adjacent to p_1, q_1, t forces the fourth color on $G(2)$ (or on $G^*(2)$ if $n = 1$). So we may assume that $c(q_1) = 1$ (see Fig. 17(b)).

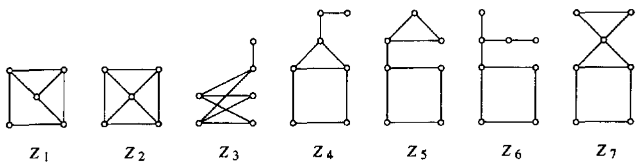

Fig. 13. The minor component of some on-line 4-critical graphs containing G_5 as major component.

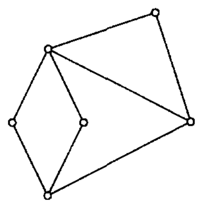

Fig. 14. An on-line 4-critical graph for C_4RFF.

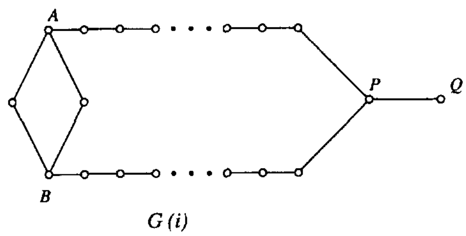

Fig. 15. The i-kite $G(i)$: the paths AP and BP have $\lfloor i/2 \rfloor$ and $\lceil i/2 \rceil$ inner vertices.

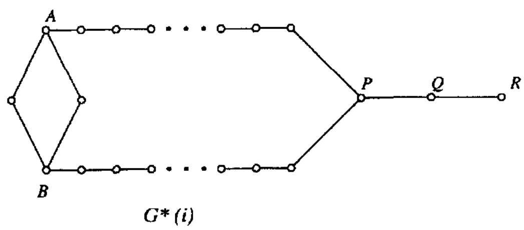

Fig. 16. The 'extended' i-kite $G^*(i)$.

Continue with q_2 adjacent to q_1. If Painter uses $c(q_2)=2$ then the fourth color is forced on $G(3)$. Otherwise p_2 is given adjacent to p_1 and Painter must color it with 1 to avoid four colors on $G(4)$. Continuing this way, Drawer can force the paths $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ to be colored with alternating colors 1 and 3. Then the last vertex is added (marked with $*$ on Fig. 17(c)) which is adjacent to p_n, q_n, u, forming the $G^*(2n)$ component. This vertex must be colored with a fourth color (see Fig. 17(c)) completing the forcing strategy. We mention that when forcing the fourth

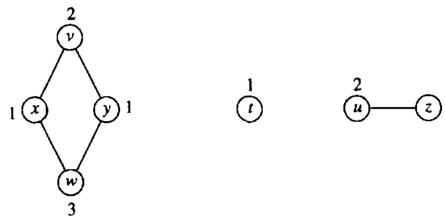

Fig. 17(a). Initialization of forcing.

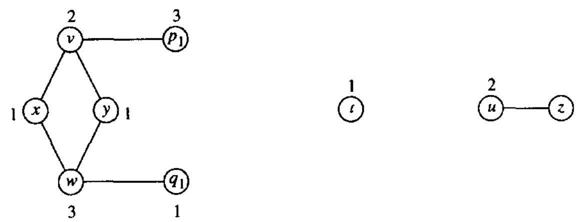

Fig. 17(b). Start of forcing.

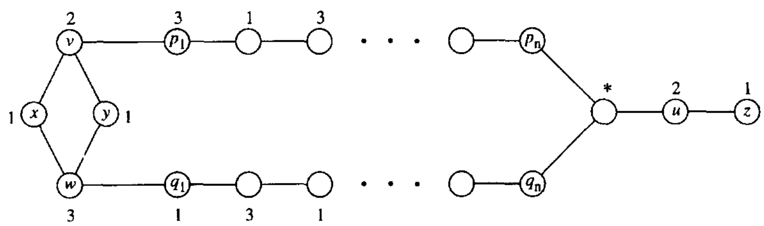

Fig. 17(c). End of forcing.

color in one component the other components (there are at least two) have enough room for putting the possibly remaining edge and two isolated vertices in them.

We finish the proof by giving on-line 3-coloring algorithms $A(i)$ for proper subgraphs of $K(n)$. Let $T(j)$ denote the following graph. Take two distinct vertices (will be called endpoints) and join them with disjoint paths of length $\lfloor j/2 \rfloor$ and $\lceil j/2 \rceil$, respectively, to opposite vertices of a C_4. (Actually $T(j)$ is obtained from $G(j)$ by deleting its tail.) Set $A(0) = \text{FF}$, $A(1) = C_4\text{RFF}$. For $2 \leqslant i \leqslant 2n$ the algorithm $A(i)$ is a refinement of C_4RFF. We apply the following special rules if the current component is isomorphic to $T(j)$ for some j and the current vertex is an endpoint of it and one of the following conditions holds.

— $j = 1$ and the color of the other endpoint is available for the current vertex. Color it by that color.

— $1 < j < i$ and the other endpoint is colored by 3. Avoid color 2.

— $1 < j = i < 2n$ and the other endpoint is colored by 3. Use color 2.

With this definition of the $A(i)$ algorithms, for $x \in V(G(i))$ ($x \in V(G^*(i))$ if $i = 2n$), the graph $K(n) - x$ is 3-colored by $A(i)$ because $A(i)$ uses four colors only on the

$G(i)$ component of $K(n)$ and after the removal of x from $V(K(n))$ no subgraph is isomorphic to $G(i)$. □

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