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A Class of Edge Critical 4-Chromatic Graphs

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Abstract. We consider several constructions of edge critical 4-chromatic graphs which can be written as the union of a bipartite graph and a matching. In particular we construct such a graph G with each of the following properties: G can be contracted to a given critical 4-chromatic graph; for each $n \geq 7$, G has n vertices and three matching edges $\left(\text{it is also shown that such graphs must have at least } \frac{8n}{5} \text{ edges}\right)$; G has arbitrary large girth.

1. Introduction

Any 4-chromatic graph can be written as the union of two bipartite graphs. In this paper we consider the more restricted family of 4-chromatic graphs, those which can be written as the union of a bipartite graph and a matching. In general $B + M$ will denote such a 4-chromatic graph, but in rare instances (when the usage causes no confusion) the $B + M$ graph may have smaller chromatic number. The simplest example of a 4-chromatic $B + M$ graph is K_4, the complete graph on four vertices, and this graph is also critical. In this paper we use the term critical in the stronger sense (sometimes called edge critical, introduced by Dirac): a graph is *critical* if the removal of any edge decreases its chromatic number. Since in this paper we mainly consider critical 4-chromatic graphs, we refer to them simply as *critical graphs*. The senior author asked for a characterization of critical $B + M$ graphs in [4]. The constructions of this paper indicate that such a characterization is unlikely since the family of these graphs is complicated.

The construction in Section 2 shows that any critical graph is a contraction of a suitable critical $B + M$ graph (Theorem 1).

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In Section 3 critical $B + M$ graphs are constructed with just three edges in M (Theorem 2). We shall call these graphs $B + 3$ *graphs*, in general a $B + m$ *graph* is a $B + M$ graph with m edges in M. Related constructions, for critical graphs which can be written as the union of a bipartite graph and a *triangle*, is due to Gallai. Further results of this kind have been proved by Toft and Nielsen in [9]. Those constructions can be generalized for k-critical graphs. Further constructions have been given by Tuza and Rödl in [12].

It seems that even critical $B + 3$ graphs are complicated. For example, we could not determine the minimum number of edges in such graphs on n vertices. The lower bound $\frac{8n}{5}$ (Theorem 3) is a slight improvement over Gallai's lower bound $\frac{20n}{13}$([7]) valid for arbitrary critical graphs on n vertices. There are no triangle free critical $B + 3$ graphs but there are infinitely many critical $B + 4$ graphs (Theorem 4).

In Section 4 we shall construct critical $B + M$ graphs with arbitrary large girth (Theorem 5) and show that $|M|$ must grow with the girth of the graph (Theorem 6).

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2. Critical $B + M$ Graphs Contractible to a Given Graph

One possibility for constructing critical $B + M$ graphs is to start from an arbitrary critical graph G and transform it to a matching, G^*, as follows. For each vertex v of G let I_v be an independent set of cardinality $d_G(v)$ (the degree of v in G) so that these independent sets are pairwise disjoint. For each edge uv of G place an edge between I_u and I_v so that these edges are pairwise disjoint. In this way G is transformed into a perfect matching G^*. Then G^{**} is defined by adding two new vertices x_v and y_v to each set I_v joining them to each other and to all vertices of I_v.

Theorem 1. *If G is a critical graph then G^{**} is a critical $B + M$ graph. Moreover G is a contraction of G^{**}.*

Proof. The second statement is clear from the definition of G^{**}; contracting each set $H_v = \{I_v \cup x_v \cup y_v\}$ into a vertex one obtains a graph isomorphic to G. It is also clear that G^{**} is a $B + M$ graph; the edges of G^* together with the edges (x_v, y_v) from a perfect matching of G^{**} and its removal leaves a graph which is the union of vertex disjoint complete bipartite graphs.

Assume G^{**} has a proper coloring χ with three colors. Each H_v must be colored so that I_v is monochromatic, so χ defines a 3-coloring on G which implies there are adjacent vertices v, w of G colored with the same color. Since there is an edge between I_v and I_w, χ is not a proper coloring. This contradiction shows that G^{**} is not 3-colorable.

To see that the removal of any edge of G^{**} leaves a 3-colorable graph, a slightly different argument is used for the three types of edges. If an edge between I_v and I_w

is removed then a proper 3-coloring of $G\backslash(v, w)$ can be transferred to G^{**} as a proper 3-coloring in an obvious way.

Assume that an edge (x_v, z) is removed where $z \in I_v$. Then there is a unique w such that z is adjacent to a vertex in I_w. Transfer the proper 3-coloring of $G\backslash(v, w)$ to G^*, without loss of generality all vertices of I_v and I_w are colored with color 1. Recolor z with color 2, color x_v with color 2 and color y_v with color 3. For each $t \neq v$, color the pair x_t, y_t with the pair of colors from $\{1, 2, 3\}$ not used on I_t. The argument for the removal of (y_v, z) is clearly the same.

Finally, if the edge (x_v, y_v) is removed for some $v \in V(G)$, remove the vertex v from G. Transfer the proper 3-coloring of $G\backslash v$ to $G^*\backslash I_v$. For $w \neq v$, as above, it is obvious how to extend this coloring to the pair x_w, y_w. To finish the coloring, color both x_v and y_v with color 1, then color the vertices of I_v greedily (with colors 2 or 3). Since each vertex of I_v is adjacent to precisely one vertex of $V(G^{**})\backslash H_v$, a proper 3-coloring of G^{**} is obtained. □

The smallest G^{**} comes from Theorem 1 when $G = K_4$. Then G^{**} has 20 vertices and 34 edges. In general, if G has n vertices and m edges, G^{**} has $2n + 2m$ vertices and $5m + n$ edges.

3. Critical $B + M$ Graphs with Small M

We begin with the easy remark that apart from K_4, critical $B + M$ graphs must contain at least three edges in their matching part, M.

Proposition 1. *If G is a $B + M$ graph with $|M| < 3$ then either $K_4 \subseteq G$ or G is 3-colorable.*

Proof. The subgraph induced by M is either a K_4 or contains $|M|$ independent vertices which can be colored by color 3. The rest of the vertices induce a bipartite graph. □

Call a triple T of distinct vertices in a 3-colorable graph G a *trichromatic triple of* G if the vertices of T are colored with three distinct colors in any proper 3-coloring of G. A 3-colorable graph G is called a *starter on* T if T is a trichromatic triple of G but T is not a trichromatic triple of $G - e$ for all $e \in E(G)$. A *tail* T_k^* is a graph of order $3k + 1$ defined as follows. The vertices of T_k^* are partitioned into k triples $T = T_0, T_1, \ldots, T_{k-1}$ and a single vertex x. The set T is called the *attachment* of the tail and the vertex x is called the *end* of the tail. The edges of T_k^* form $k - 1$ edge disjoint six-cycles each forming a bipartite graph with partite classes T_{i-1}, T_i $(1 \leq i \leq k - 1)$ plus three edges from x to T_{k-1}. The edges within the attachment are not specified so there are four types of tails T_k^* depending upon the number of edges in $T(0, 1, 2, 3)$.

Lemma. *Assume that G is a starter on T and T_k^* is a tail with attachment T, $V(G) \cap V(T_k^*) = T$. Then $H = G \cup T_k^*$ is a critical* (4-*chromatic*) *graph.*

Proof. Assume there is a proper 3-coloring of H, then T is colored with 3 distinct

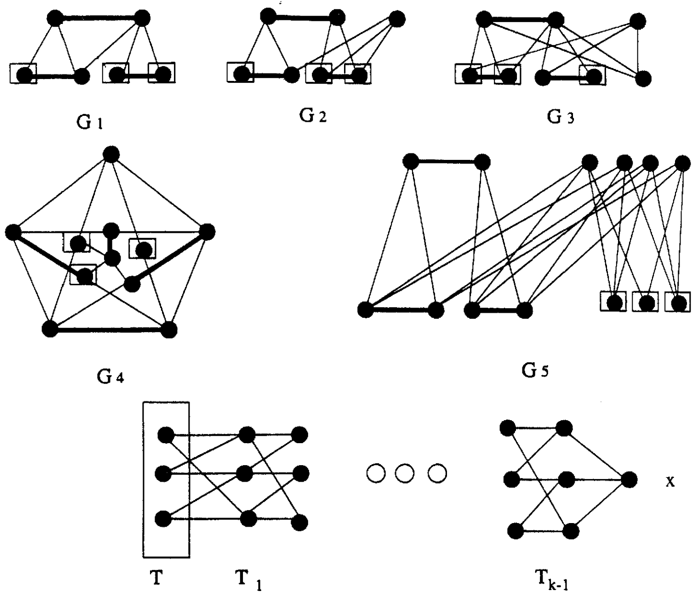

Fig. 1. Starters and Tail

colors which is propagated along the triples of the tail and gives a contradiction at the end of the tail. On the other hand, removing an edge e from H which is in G allows one to 3-color G so that T is colored with at most two colors. This coloring is obviously extendible to a proper 3-coloring of H. Similarly, if e is an edge of the tail then starting from an arbitrary proper 3-coloring of G, the missing edge of the tail allows one to extend the 3-coloring of G to a proper 3-coloring of $H - e$. $\square$

Theorem 2. *For each $n > 6$ there are critical $B + 3$ graphs with n vertices.*

Proof. Applying the lemma, it is enough to find starters on six, seven and eight vertices which become bipartite after attaching a tail and deleting the three independent edges. It is not difficult to check that G_1, G_2, G_3 are such starters. (See Fig. 1, the trichromatic triples of the starters are marked with boxes and the three edges of M are the heavy edges.) $\square$

Theorem 3. *A critical $B + 3$-graph with n vertices must have at least $\frac{8n-2}{5}$ edges.*

Proof. Let G be a critical $B + 3$ graph of order n, let L denote the set of low points of G, i.e. the set of vertices of degree 3. Set $e_1 = |\{(x, y) \in E(G) | x, y \in L\}|$, $e_2 = |\{(x, y) \in E(G) | x \in L, y \notin L\}|$, and $e_3 = |\{(x, y) \in E(G) | x, y \notin L\}|$.

From a theorem of Gallai [7] the blocks of $G[L]$ are complete graphs or odd cycles ($G[L]$ denotes the subgraph induced by L in G). In our case (unless $G = K_4$) all blocks must be odd cycles or edges. Moreover, a $B + 3$ graph can not contain more than three disjoint odd cycles. Therefore it is possible to make $G[L]$ acyclic by the removal of at most three edges. This gives

(1) $$e_1 \leq |L| + 2.$$

The degree counts give

(2) $$3|L| = 2e_1 + e_2$$

and also

(3) $$4(n - |L|) \leq 2e_3 + e_2.$$

Combining (1) and (2), and (2) and (3), one obtains two lower bounds on the size of G:

$$|E(G)| \geq 2|L| - 2, \qquad 2|E(G)| \geq 4n - |L|.$$

The theorem follows by eliminating $|L|$ from the two inequalities. □

Theorem 4. *There are no triangle free critical $B + 3$ graphs but there are infinitely many triangle free critical $B + 4$ graphs.*

Proof. Assume that G is a triangle free critical $B + 3$ graph with $G = B \cup M$ where $|M| = 3$. Since G is triangle free, the six vertex graph induced by M has an independent set I of three vertices (Ramsey's theorem). Color I with color 3 and the rest with 1 and 2 according to the bipartition of $V(G)\backslash I$. Therefore G is 3-colorable. The second part of the theorem follows by checking that the lemma is applicable using starter G_4 shown in Figure 1. □

4. Critical $B + M$ Graphs with Large Girth

Theorem 5. *The girth of critical $B + M$ graphs can be arbitrary large.*

Proof. The proof will use uniform hypergraphs with arbitrary large girth and chromatic number (see [6] for existence, [8] and [10] for construction) as building blocks and "glue" them by factors like B. Descartes constructed graphs of girth six with large chromatic number in [2].

Step 1. First a 3-chromatic $B + M$ graph G_1 of large girth is constructed with B's partite classes being X_1, X_2, so that for any proper 3-coloring of G_1, *either* X_1 *or* X_2 must receive all three colors. This construction is easy: for k congruent 3 mod 4, $G_1(k)$ is defined by taking two vertex disjoint k-cycles C_1, C_2 with an edge between them. The arrangement of $G_1(k)$ in the partite classes X_1, X_2 is shown on Fig. 2. Checking the required property is left to the reader. For brevity, the parameter k is omitted and the graph constructed will be referred to as $G_1 = G_1(X_1, X_2)$.

Step 2. The construction of Step 1 is strengthened to show that there is a 3-chromatic $B + M$ graph G_2 of large girth so that a *specified* partite class of B, say X_1, must receive all three colors in any proper 3-coloring of G_2. Perhaps there is a

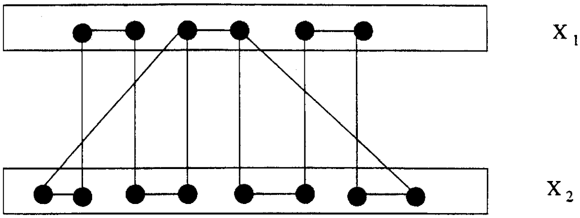

Fig. 2

simple direct construction for G_2 (as in Step 1) but we choose to use a more complex construction since this is needed as well in Step 3.

Let $G_1(X_1, X_2)$ be the graph from Step 1 and let n denote the number of vertices in X_2. Let $\mathscr{H}$ be an n-uniform 3-chromatic hypergraph without small cycles. With each edge E of $\mathscr{H}$ associate a copy of $G_1(X_1, X_2)$ and place a matching between E and X_2. For distinct edges of $\mathscr{H}$, vertex disjoint copies of G_1 are used. The graph G_2 is the graph defined as the union of these copies so that the new X_1 is the union of the old X_1-s of the copies and the vertex set of $\mathscr{H}$. Also, the new X_2 is the union of the old X_2-s.

It is clear from the construction that G_2 is a $B + M$ graph. To see that G_2 has no small cycles (assuming that this is true for G_1) notice that a cycle is either a cycle of a copy of G_1 or it is a cycle which connects paths of these copies through the matching edges. In this case it defines a cycle in the hypergraph $\mathscr{H}$ but $\mathscr{H}$ also has large girth. In fact, if $\mathscr{H}$ has girth k than such a cycle has length at least $3k$. Therefore it is enough to require that $\mathscr{H}$ has girth k to ensure that G_2 has girth k.

We claim that X_1 receives all three colors in any proper 3-coloring of G_2. Let χ be a proper 3-coloring of G_2. If only two colors are used on X_1 then the vertex set of $\mathscr{H}$ is 2-colored so $\mathscr{H}$ has a monochromatic hyperedge E. This means that the X_2-part is two-colored in the copy of G_1 associated with E. Then, by Step 1, the X_1 part of this copy must receive all three colors. This proves the claim.

Step 3. This step is similar to Step 2. Assume that G_2 is a graph constructed in Step 2, and set $n = |X_1|$. Let $\mathscr{H}$ be an n-uniform 4-chromatic hypergraph of large girth. A copy of G_2 is associated with each edge of $\mathscr{H}$ so that these copies are vertex disjoint and a matching is placed between the X_1 part of the copy and the edge of $\mathscr{H}$. This defines G_3, its X_1 part is the union of the X_1 parts of the copies of G_2; its X_2 part is the union of the X_2 parts of the copies and the vertex set of $\mathscr{H}$.

The graph G_3 is clearly a $B + M$ graph (built from many copies of $G_1(k)$). Like above, it has girth k if the girth of $\mathscr{H}$ is selected to be k. Assume χ is a proper 3-coloring of the vertex set of G_3. Then χ colors the vertices of $\mathscr{H}$ as well which implies that there is a monochromatic hyperedge E of $\mathscr{H}$. Consider the X_1-part of the copy of G_2 associated with E. On one hand, by construction in Step 2, X_1 receives all three colors, on the other hand X_1 is matched with the monochromatic vertex set E. This is a contradiction showing that G_3 is a 4-chromatic graph. The proof is finished by noting that a critical subgraph of G_3 is a critical $B + M$ graph. □

The next theorem gives a lower bound on the size of the matching part in a critical $B + M$ graph in terms of the girth. Probably much better bounds can be given, but we selected one which has a short proof using a Ramsey type result.

Theorem 6. *A critical $B + M$ graph of girth g satisfies $|M| \geq g - 1$.*

Proof. Assume G is a critical $B + M$ graph such that $|M| = m \leq g - 2$. A Ramsey type result of Erdős et al. ([5], Theorem 3) says that any graph on $2m$ vertices contains either a cycle of length at most $m + 1$ or an independent set of m vertices. Applying this to $G[M]$, the former possibility is excluded because $m + 1 \leq g - 1$. Therefore $G[M]$ has an independent set I of m vertices. The removal of I leaves G bipartite so G is 3-colorable. This contradicts the assumption that G is critical and proves the theorem. □

5. Conclusion

Perhaps the reader is convinced that critical $B + M$ graphs are complicated. In fact, even critical $B + 3$ graphs seem to be complicated. For example, the starter G_5 of Figure 1 is more complicated than the others.

Among the well-known 4-critical graphs, the Toft graph (two odd cycles factored into the partite classes of a complete bipartite graph, [11]) is a $B + M$, the Grötsch graph is a $B + 4$. The Grünbaum graph [3] is an explicit example of a critical $B + M$ graph of girth five. The simplest critical graph which is not a $B + M$ graph is the wheel, an odd cycle of length at least five with a further vertex adjacent to all vertices of the cycle. Another example can be obtained by selecting a triangle as a starter (this is used in [12]).

The minimum number of edges in a 4-critical graph of n vertices, $f(n)$, is conjectured to be asymptotic to $\frac{5n}{3}$. T. Gallai improved the lower bound of $f(n)$ by increasing the trivial $\frac{3n}{2}$ bound to $\frac{20n}{13}$. There is an easy example of a 4-critical $B + M$ graph with $\frac{5n}{3}$ edges: a chain of $K_4 - e$ graphs (complete graph minus an edge), formed by identifying degree 2 vertices of consecutive members of the chain and adding an edge between the ends of the chain. A graph of Abbott, Hare and Zhou ([1], Theorem 3) is an example of a critical triangle free $B + M$ graph with number of edges asymptotic to $\frac{5n}{3}$.

We know that one has to be careful with conjectures in this area. That is why we only suspect that 4-critical $B + 3$ graphs of n vertices must have at least $2n$ edges asymptotically and dare to conjecture only that they have significantly more than $\frac{5n}{3}$ edges. However, it is safe to propose the subconjecture that a critical $B + M$ graph must have at least $\frac{5n}{3}$ edges.

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