# An Edge Coloring Problem for Graph Products

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**ABSTRACT**

The edges of the Cartesian product of graphs $G \times H$ are to be colored with the condition that all rectangles, i.e., $K_2 \times K_2$ subgraphs, must be colored with four distinct colors. The minimum number of colors in such colorings is determined for all pairs of graphs except when $G$ is 5-chromatic and $H$ is 4- or 5-chromatic. 

## 1. INTRODUCTION

A *rectangle* in the Cartesian product $G \times H$ of two graphs is a four-cycle in the form $K_2 \times K_2$. In this paper the term *good coloring* is used for edge colorings of $G \times H$ such that all rectangles are colored with four distinct colors. We determine $rb(G, H)$, the minimum

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number of colors needed for a good coloring of $G \times H$ ('$rb$' stands for 'rainbow'). We shall prove that (with two possible exceptions) $rb(G,H)$ depends only on $\chi(G)$ and $\chi(H)$, where $\chi$ denotes the chromatic number. Precisely, $rb(G,H)$ is the maximum of $\chi(G)$ and $\chi(H)$, if this maximum is at least 6 (Corollary 1). In two cases, when $\chi(G) = 4, \chi(H) = 5$ and when $\chi(G) = \chi(H) = 5$, we have not determined whether $rb(G,H)$ is 5 or 6. The other 'small' cases are settled (Theorem 2).

The problem arose from studying edge colorings of graphs where all four-cycles must be colored with four distinct colors. Such a coloring was called *rainbow* in [2] where product graphs, in particular the cube, had been considered for rainbow colorings with as few colors as possible. (The term 'rainbow' had been used in a similar sense in so called 'Anti-Ramsey' Theory where instead of monochromatic structures heterogenous structures are investigated.) Our results about good colorings can be interpreted as results about rainbow coloring if neither $G$ nor $H$ contains a four-cycle. Indeed, in this case the only four-cycles of $G \times H$ are rectangles and good colorings are exactly the rainbow colorings.

It turns out that most results depend on the minimum number of colors needed for a good coloring of $K_m \times K_n$. This number is denoted by $rb(m,n)$, and $rb(n,n)$ is abbreviated as $rb(n)$. Using complementary weak decompositions of graphs the values of $rb(m,n)$ will be determined for all values of $m$ and $n$. The main result is that $rb(n) = n$ for $n \geq 6$ (Theorem 1).

Although it seems to be a formal similarity, it is worth noting that a lemma of Shelah [7] is formulated in [4] as a similar edge-coloring problem of $K_m \times K_n$ with a weaker condition on the rectangles: rectangles can not be colored with two alternating colors. In this case it seems to be very difficult to find the order of magnitude of the minimum number of colors (some results are in [3], [5], [6].

## 2. RESULTS

**Proposition 1.** $rb(G,H) \leq rb(\chi(G), \chi(H))$.

***Proof.*** Assume that $V(G)$ is partitioned into $k$ independent sets $A_i$ and $V(H)$ is partitioned into $l$ independent sets $B_i$, where $k = \chi(G)$ and $l = \chi(H)$. Contract each set $A_i \times B_j$ in $G \times H$ into a single vertex $v_{ij}$ and view this as $M = K_k \times K_l$ by adding all horizontal and vertical edges. There is a good coloring of $M$ with $rb(k,l)$ colors. Transfer this coloring to $G \times H$ by coloring each edge between $A_i \times B_j$ and $A_i \times B_t$ with the color of $v_{ij}v_{it}$ in $M$, similarly by coloring each edge between $A_i \times B_j$ and $A_t \times B_j$ with the color of $v_{ij}v_{tj}$. This is a good coloring of $G \times H$. ∎

**Proposition 2.** $rb(G,H) \geq \max(\chi(G), \chi(H))$.

***Proof.*** Assume that $rb(G,H) = t$ and let $\alpha$ be a good coloring of $G \times H$ with $t$ colors. Fix an arbitrary edge $e$ of $G$, and for each vertex $v$ in $H$ color $v$ with $\alpha(e \times v)$. Since $\alpha$ is a good coloring of $G \times H$, we obtain a proper coloring of the graph $H$ with at most $t$ colors. Therefore $\chi(H) \leq t$. A similar argument shows that $\chi(G) \leq t$. ∎

In preparation of the main result, Theorem 1, we define a decomposition for complete graphs which resembles the well studied concept of cyclic decomposition (see for example in [1]). The vertex set of $K_n$ will be $\{0, 1, \ldots, n-1\} = [n]$. On subsets of $[n]$ we shall use mod $n$ arithmetic, and for $X \subseteq [n]$, $-X$ is defined as $\{-x : x \in X\}$. A decomposition of $K_n$ into $n$ subgraphs $G_0, G_1, \ldots, G_{n-1}$ is called *weak* if $V(G_i) = V(G_0) + i$ for each $i \in [n]$ and each edge of $K_n$ is in precisely one $G_i$. This is much weaker notion than a cyclic decomposition, for example $G_0$ can be $K_n$ and all other $G_i$'s are graphs with vertex set $[n]$ and with no edges. Two weak decompositions of $K_n, \{G_i\}$ and $\{H_i\}$ are called *complementary* if $(-V(G_0)) \cap V(H_0) = \emptyset$. For certain values of $n$ there exist complementary cyclic decompositions of $K_n$ into complete subgraphs. For example, if $n = 7, V(G_0) = \{0, 1, 3\}$ and $V(H_0) = \{2, 4, 5\}$ with $G_0$ and $H_0$ both triangles defines such an example. Under the notion of weak decomposition, complementary decompositions exist for any large $n$ as shown in the next lemma.

**Lemma 1.** For each $n \geq 6$ there exist weak decompositions of $K_n$ which are complementary.

***Proof.*** Set $m = \lfloor n/2 \rfloor$ and define

$$A = \{0, 1, \ldots, m-3, m-2, m\}.$$

For odd $n$, define $B$ as

$$B = \{2, 3, \ldots, m-1, m, m+2\},$$

and for even $n$, define $B$ as

$$B = \{1, 2, \ldots, m-2, m-1, m+1\}.$$

In both cases $-A \cap B = \emptyset$ holds. Therefore defining $G_0$ and $H_0$ as complete graphs with vertex sets $A$ and $B$, respectively, the condition of complementary decompositions is satisfied. Furthermore, if $n \geq 6$ (i.e., $m \geq 3$), the edges of the complete graphs $G_0 + i$ and $H_0 + i$ both cover the edges of $K_n$ on $[n]$ because the differences of elements in both $A$ and $B$ give all elements of $[n]$. Therefore recursively defining the edge set $E_i$ for $i = 1, 2, \ldots, n-1$ of the graph $G_i$ as

$$E_{i+1} = \left\{ \bigcup pq : p, q \in V(G_0) + i, pq \notin E_j, 0 \leq j \leq i \right\}$$

(with $E_0 = E(G_0)$), the graphs $G_i$ form a weak decomposition. The graphs $H_i$ can be defined similarly, writing $H$ instead of $G$. ∎

**Theorem 1.** $rb(n) = n$ if $n \geq 6$.

***Proof.*** Since $r(n) \geq n$ from Proposition 2, we have to give a good coloring of $M = K_n \times K_n$ with $n$ colors. Represent the vertex set of $M$ as a matrix $a_{i,j}, i, j \in [n]$. Let $G_i$ and $H_i$ be two weak complementary decompositions of $K_n$. Their existence is ensured by the lemma.

Define the coloring of the edges of $M$ with $n$ colors as follows. For $k, i \in [n]$, edges of color $k$ in row $i$ of $M$ are defined by the image of the graph $G_{i+k}$ under the isomorphism

$$x \to a_{i,x} : x \in V(G_{i+k})$$

Also, for $k, j \in [n]$, edges of color $k$ in column $j$ of $M$ are defined by the image of the graph $H_{j-k}$ under the isomorphism

$$y \to a_{y,j} : y \in V(H_{j-k})$$

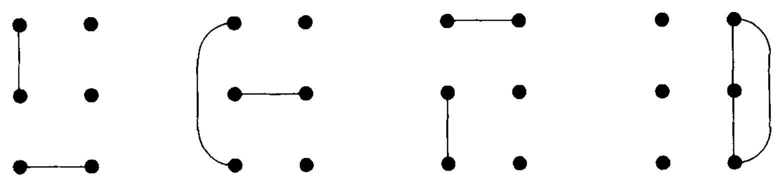



FIGURE 1.

Note that the graphs obtained from $V(G_{i+k})$ by varying $k$ form a weak decomposition so that all edges in the $i$th row (for each $i$) are colored once. By varying $i$ for each value of $k$, the weak decomposition also gives that two horizontal edges of an arbitrary rectangle are colored with distinct colors. The same argument applies to columns using that $H$-$s$ form a weak decomposition.

It remains to show that no 'corner' of some rectangle can have both incident edges of the same color. If not, there is a 'corner' of some rectangle with both incident edges of color $k$, for some $k \in [n]$. This means that for this value $k$ and for some $i, j \in [n]$ the images of the two maps coincide. Therefore, for some $x \in V(G_{i+k}) = V(G_0) + i + k$ and for some $y \in V(H_{j-k}) = V(H_0) + j - k, i = y$ and $j = x$. This means that $i = y_1 + j - k$ and $j = x_1 + i + k$ for some $x_1 \in V(G_0), y_1 \in V(H_0)$. Adding these equations we get $x_1 + y_1 = 0$, contradicting the fact that the decompositions are complementary. ∎

**Corollary 1.** $rb(G,H) = m$ if $\max\{\chi(G), \chi(H)\} = m \geq 6$.

***Proof.*** From Proposition 1 and from Theorem 1

$$rb(G,H) \leq rb(\chi(G), \chi(H)) \leq rb(m,m) = rb(m) = m.$$

On the other hand, from Proposition 2,

$$rb(G,H) \geq \max\{\chi(G), \chi(H)\} = m.$$

∎

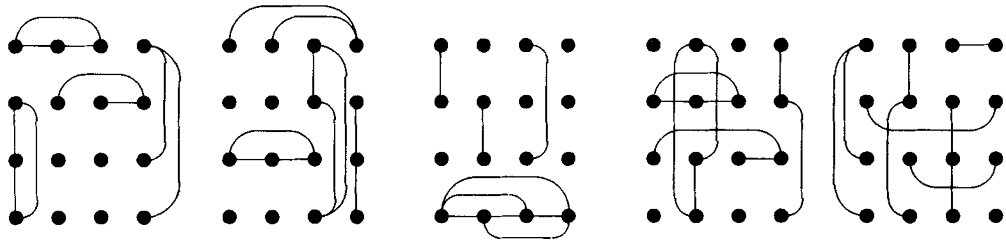
**Proposition 3.** $rb(G,H) = 4$ if $\chi(G) = 2$ and $\chi(H)$ is either 2 or 3.

***Proof.*** Since $G \times H$ contains at least one rectangle, $rb(G,H) \geq 4$. On the other hand, $rb(2,3) \leq 4$ is shown by the coloring of Figure 1. From Proposition 1 we have, $rb(G,H) \leq rb(\chi(G), \chi(H)) \leq rb(2,3) \leq 4$. ∎

**Theorem 2.** Let $G, H$ be three- or four-chromatic graphs. Then $rb(G,H) = 5$.

***Proof.*** Figure 2 shows that $rb(4) \leq 5$. Therefore, using Proposition 1 we have, $rb(G,H) \leq rb(\chi(G), \chi(H)) \leq rb(4,4) = rb(4) \leq 5$.

To prove the reverse inequality, assume that $G \times H$ has a good coloring $\alpha$ with at most 4 colors. Since both $G$ and $H$ are at least 3-chromatic, they contain odd cycles. Let $C_p$ and $C_q$ be odd cycles of $G$ and $H$, both without diagonals. Clearly, $\alpha$ is a good coloring on the subgraph $C = C_p \times C_q \subseteq G \times H$. For $i = 1, 2, 3, 4$ let $M_i$ denote the set of edges



Color 1 Color 2 Color 3 Color 4 Color 5

FIGURE 2.

in $C$ colored with color $i$. Without loss of generality,

$$|M_1| \leq |M_2| \leq |M_3| \leq |M_4|.$$

Since $C$ has $2pq$ edges,

$$|M_1| \leq \left\lfloor \frac{pq}{2} \right\rfloor.$$

Observe that the edge set $F = E(C) \setminus M_1$ does not form rectangles in $C$ because $\alpha$ is a good coloring of $C$. Since $C$ has $pq$ rectangles and each edge of $C$ is in precisely two rectangles, $F$ determines at least

$$pq - 2|M_1| \geq pq - 2\left\lfloor \frac{pq}{2} \right\rfloor$$

rectangles. Since both $p$ and $q$ are odd, the right hand side of the above inequality is equal to one. This contradiction proves the theorem. ∎

**Theorem 3.** $rb(4,5) = rb(5) = 6$.

***Proof.*** On one hand, $rb(5) \leq rb(6) = 6$ follows from Theorem 1. To prove the reverse inequality, assume that there exists a good coloring $\alpha$ of $M = K_4 \times K_5$ with at most 5 colors. Since $M$ has 70 edges, one color class, say color class 1 contains at least 14 edges. Among these, at most 6 edges are 'vertical', so one can select a set $F$ of 8 horizontal edges each colored with color 1. We say that $f \in F$ *spans* the columns of its endpoints. Consider the following property.

**Property** (*)**:** the edges of $F$ in any two rows of $M$ span at most four columns.

If this were not the case, then at least two of the five vertical edges determined by the pair of rows spanning all five columns would receive the same color.

We claim the property (*) must be violated, which will prove the theorem. Consider the row of $M$ containing the largest number of edges from $F$, say it is row 1. Assume that edges of $F$ in row 1 span $t$ columns. From the choice of row 1, $t \geq 3$, and we know that $t \leq 4$.

*Case 1.* $t = 4$. There are at most six edges of $F$ with both endpoints in the spanned $t$ columns. Since $|F| > 6$, there is an edge of $F$ in row $j$ which spans the column not spanned by the edges of $F$ in row 1. Then row 1 and row $j$ spans all five columns of $M$, contradicting (*).

*Case 2.* $t = 3$. By the definition of $t, F$ spans at most 3 columns in each row. We may assume columns 1, 2, 3 are spanned by $F$ in row 1. Since there are at most three edges of $F$ with both endpoints in columns 1, 2, 3, there must be a set $T \subset F$ of five edges in rows 2, 3, 4 such that each of them spans either column 4 or column 5. No edge of $T$ spans both columns 4 and 5 otherwise (*) is violated. For the same reason no two edges of $T$ in the same row span both columns 4 and 5. Thus, in each row, edges of $T$ span at most one of the columns 4 and 5. Since $T$ spans at most three columns in each row, $T$ appears in $1 - 2 - 2$ distribution in rows 2, 3, 4. Without loss of generality rows 2 and 3 contain two edges of $T$. Since $\alpha$ is a good coloring, the edges of $T$ in rows 2 and 3 form stars with centers in different columns (one of them has its center in column 4, the other has its center in column 5). The endpoints of these stars span the same pair of columns (among columns 1, 2, 3) otherwise (*) is violated. Since $\alpha$ is a good coloring, the edge of $T$ in row 4 and one of the two stars span all columns, violating (*). ∎

We close the paper with the two (related) open problems: is it true that $rb(G, H) = 6$ if $\chi(G)$ is either 4 or 5 and $\chi(H) = 5$?

## References


[1] J. Bosák, *Decompositions of graphs,* Vol. 47 in Mathematics and Its Applications, Kluwer Academic Publishers (1990).

[2] R. J. Faudree, A. Gyárfás, L. Lesniak, and R. H. Schelp, Rainbow coloring of the cube, *J. Graph Theory* **17** (1993), 607–612.

[3] R. J. Faudree, A. Gyárfás, and T. Szőnyi, Projective spaces and colorings of $K_m \times K_n$, *Coll. Math. Soc. J. Bolyai* **60,** Sets Graphs and Numbers (1991), 273–278.

[4] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory,* 2nd edition, Wiley-Interscience, New York (1990).

[5] A. Gyárfás, On a Ramsey type problem of Shelah, Extremal Problems for Finite Sets, *Bolyai Soc. Math. Studies* **3,** (1994), 283–287.

[6] K. Heinrich, Coloring the edges of $K_n \times K_n$, *J. Graph Theory* **14** (1990), 575–583.

[7] S. Shelah, Primitive recursive bounds for van der Waerden numbers, *J. Amer. Math. Soc.* **1** (1989), 683–697.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