Endpoint Extendable Paths in Tournaments

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ABSTRACT

Let $s(n)$ be the threshold for which each directed path of order smaller than $s(n)$ is extendible from one of its endpoints in some tournament T_n. It is shown that $s(n)$ is asymptotic to $3n/4$, with an error term at most 3 for infinitely many n. There are six tournaments with $s(n) = n$.

It is well-known that every tournament has a Hamiltonian path [8]. In fact, the vertex set of any directed path P_k is a subset of the vertex set of a P_{k+1} in every tournament of at least $k+1$ vertices. So the order of any maximum or maximal path of an n-vertex tournament is always n. However, if maximal paths are defined as paths which are not extendible from their endpoints, nontrivial problems arise naturally. These maximal paths can be interpreted as walks of Fred and Buck who are at a vertex of a tournament and explore unexplored vertices as long as they can, with the rule that Fred always goes forward, Buck always goes backward on the arcs. The undirected version, (when Fred and Buck explore an undirected graph) has been studied in several papers including [2–6, 9]. The problem about the possible lengths of the maximal paths of an undirected graph was introduced to us by Mike Jacobson et al. (as the path spectrum problem) and it was discovered later that

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Carsten Thomassen had considered similar type problems. Those former investigations motivated the present paper about the tournament version.

To have formal definitions, a *path* or a *cycle* of a tournament is always assumed to be directed, we use the notation P_k, C_k, if they have k vertices (for a cycle, $k \geq 3$). The vertex set of a tournament T (path P) is denoted by $V(T)(V(P))$. The number of vertices of a tournament T (path P) is referred to as the *order* of $T(P)$. Tournaments or paths of order 1 are called *trivial.* The notation T_n stands for an arbitrary tournament of order n. We use the notation (x, y) for the edges (arcs) and $(x, y) \in T$ indicates that (x, y) is an edge of T. The notation $N^+(x)$ and $N^-(x)$ is used for the set of vertices $\{y|(x, y) \in T\}, \{y|(y, x) \in T\}$ respectively; their cardinalities will be referred as the *outdegree* and *indegree* of x in the tournament T. The *sum* of vertex disjoint tournaments $M_1, M_2, \ldots, M_t$ is the tournament defined by adding all arcs pointing from smaller index tournaments to larger index tournaments. Any tournament can be uniquely written as the sum of strong tournaments (see, for example [7]).

A path $P = x_1, x_2, \ldots, x_k$ of a tournament T is called *endpoint extendible* if there exists $x \in V(T) \setminus V(P)$ such that either $(x, x_1) \in T$ or $(x_k, x) \in T$. A path is *maximal* if not endpoint extendible. A moments reflection on the well-known proof of Rédei's theorem (every tournament has a Hamiltonian path) gives

Proposition 1. If a tournament T_n has a maximal path of order $k < n$ then T_n also has a maximal path of order $k + 1$.

Proposition 1 allows one to define the threshold number for maximal paths, or simply *threshold* of $T, s(T)$ as the unique integer m for which each path of T with order smaller than m is endpoint extendible, but for all k such that $m \leq k \leq |V(T)|, T$ has a maximal path of order k.

Clearly, for any nontrivial $T_n, 2 \leq s(T_n) < n$. Tournaments with $s(T) = 2$ have obvious characterization. The other extreme case, when $s(T_n) = n$, is characterized in Theorem 1. Adapting the terminology used by Jacobson, Lehel and Kézdy [5, 6], such tournaments might be called 'scenic'—Fred and Buck always end up exploring a Hamiltonian path.

Theorem 1. $s(T_n) = n$ if and only if T_n is the sum of at most two terms selected from $\{P_1, C_3\}$. (There are six scenic tournaments.)

Proof. It is easy to see that $s(T_n) = n$ for the six tournaments described in the statement of the theorem. For the reverse implication, assume that $s(T_n) = n$. Write T_n as the sum of its irreducible (strong) components,

$$T_n = S_1 + S_2 + \cdots + S_k$$

∎

Case 1. $k = 1$, i.e., T_n is strong.

Assume that $n \geq 4$. Since a nontrivial strong T_n contains cycles of any order (between three and n), there exists $x \in V(T_n)$ such that $T_n \setminus x$ has a cycle C_{n-1}. The sets $N^+(x), N^-(x)$ define a nontrivial partition on the vertices of C_{n-1} which implies that there exists an arc (u, v) of C_{n-1} such that $u \in N^+(x), v \in N^-(x)$. The path obtained from C_{n-1} by removing the arc (u, v) is a maximal path of T_n, a contradiction. Therefore $n \leq 3$ so T_n is either P_1 or C_3.

Case 2. $k = 2$.

Assume that $s(S_1) < |S_1|$; then S_1 has a maximal path of order smaller than $|S_1|$ whose endpoint can be joined to the starting point of a Hamiltonian path of S_2. This gives a maximal path of order smaller than n, a contradiction. The assumption $s(S_2) < |S_2|$ leads to contradiction in a similar way. Therefore $s(S_i) = |S_i|$ and (referring to Case 1), S_i is P_1 or C_3 $(i = 1, 2)$.

Case 3. $k \geq 3$.

This leads to an immediate contradiction by noting that for any $x \in V(S_2)$, any Hamiltonian path of $T_n \setminus x$ is maximal. ∎

Theorem 1 implies that apart from the six scenic tournaments, every T_n has a maximal path P_{n-1}. The vertex not on P_{n-1} with the two arcs to (and from) the starting point (endpoint) of P_{n-1} is a 'skew cycle' (a cycle in which two consecutive arcs are reversed). This is formulated as

Corollary 1. Every tournament, except the six scenic tournaments, contains a spanning skew cycle.

The rest of the paper is devoted to estimate $s(n)$, *the maximum of* $s(T)$ *over all tournaments* T *of* n *vertices.* The tournaments on which the maximum is achieved are the 'most scenic' n-vertex tournaments—Fred and Buck are guaranteed to explore at least $s(n)$ vertices on them.

Theorem 1 can be used to determine $s(n)$ for small values of n.

Proposition 2. $s(i) = i$ for $i \in \{1, 2, 3, 4, 6\}, s(i) = i - 1$ for $i \in \{5, 7, 8\}$.

Proof. Because of Theorem 1, one has only to furnish examples showing that the claimed values cannot be lowered. The six scenic tournaments already given with Theorem 1 provide examples for $i = 1, 2, 3, 4, 6$. For $i = 5$ take (for example) $C_3 + T_2$, for $i = 7$ take C_3 + the strong T_4 and for $i = 8$ take C_3 + the 2-regular T_5. ∎

Theorem 2. $s(n) \leq n - \lfloor n/4 \rfloor + 2$.

Proof. A source (sink) of a tournament is a vertex which dominates (is dominated by) all other vertices of the tournament. Let T_n be a tournament and select a (maximum order) subtournament S in T_n such that S has a source a and a sink b, and the order of S is as large as possible. One way to define a large S is to select a vertex a with maximum outdegree in T_n and then select $b \in N^+(a)$ with maximum indegree in $N^+(a)$. Therefore (using the fact that a tournament of order p must contain vertices with outdegree (indegree) at least $\lceil (p-1)/2 \rceil$),

$$|S| \geq \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

Set $H = V(T_n) \setminus S$ and select a Hamiltonian path $P = (x, \ldots, y)$ in the tournament induced by H. By the choice of S, the sets $X = N^-(x) \cap S$ and $Y = N^+(y) \cap S$ are non-empty.

Case 1. $X = \{b\}$. Select $z \in Y$, start at x, follow P, from y go to z and from z go to b. This path is maximal, using at most two vertices from S ($z = b$ or $z = a$ is possible).

Case 2. $Y = \{a\}$. Select $z \in X$, start at a, go to z, go to x, follow P to the very end, y. This path is maximal, using at most two vertices from S ($z = a$ or $z = b$ is possible).

Case 3. Select $z_1 \in X \setminus \{b\}$ and $z_2 \in Y - \{a\}$ and, if possible, avoid $z_1 = z_2$. Go from a to z_1, go to x, follow P, if $z_1 = z_2$ then stop at y. Otherwise continue from y to z_2 and go to b. The path defined this way uses at most four vertices from $S(a = z_1, b = z_2, z_1 = z_2$ are possible).

In all cases a maximal path is found that uses at most four vertices from S, and so the theorem follows from the displayed inequality for $|S|$. ∎

The lower bounds for $s(n)$ come from results on random and quasirandom tournaments.

Theorem 3. For every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for all $n \geq n_0, s(n) \geq (\frac{3}{4} - \epsilon)n$. Moreover, for infinitely many $n, s(n) \geq (3n - 1)/4$.

Proof. Let $f(n)$ be defined as $\min\{\max\{|N^+(x) \cap N^-(y)|\}\}$ where the maximum is taken over all ordered pairs of distinct vertices x, y of a fixed T_n and the minimum is taken over all tournaments on n vertices. Assume that $P = (x, \ldots, y)$ is a maximal path in a tournament T of n vertices. Then

$$V(T) \setminus V(P) \subseteq N^+(x) \cap N^-(y).$$

Therefore $n - |P| \leq f(n)$, i.e., $|P| \geq n - f(n)$. This means that upper bounds on $f(n)$ provide lower bounds for $s(n)$.

For arbitrary n it is known (based on random tournaments, see [7, pp. 32–33]) that $f(n) \leq (\frac{1}{4} + \epsilon)n$ for any positive ϵ and sufficiently large n (depending on ϵ). This gives the first part of the theorem.

The second part of the theorem follows by using Paley tournaments. It is well-known that for certain properties the Paley tournaments give sharper results than the random tournaments. This is true here. One can give better upper bound for $f(n)$ if n is a prime congruent 3 modulo 4. The following result is from the book of Alon and Spencer ([1, p. 117]): if $n = 4k + 3$ is a prime then, for any two distinct vertices x, y of the Paley tournament on n vertices

$$|(N^+(x) \cap N^-(y)) \cup (N^-(x) \cap N^+(y))| = 2k + 1$$

Using this inequality and the regularity of Paley tournaments, it follows easily that

$$|N^+(x) \cap N^-(y)| \leq k + 1,$$

which implies that $f(n) \leq (n + 1)/4$ for Paley tournaments and proves the second part of the theorem. ∎

References

[1] N. Alon and J. Spencer, *The probability method,* John Wiley, New York (1992).

[2] G. Chen, R. J. Faudree, and I. Shiermeyer, Graphs with all maximal paths are odd length, preprint.

[3] G. Chen, R. J. Faudree, and L. Soltes, Path spectra for trees, preprint.

[4] M. S. Jacobson, A. E. Kézdy, E. Kubicka, G. Kubicki, J. Lehel, C. Wang, and D. B. West, The path spectrum of a graph, preprint.

[5] M. S. Jacobson, A. E. Kézdy, and J. Lehel, Scenic graphs I: Traceable graphs, submitted.

[6] M. S. Jacobson, A. E. Kézdy, and J. Lehel, Scenic Graphs II: Non-traceable graphs, preprint.

[7] J. W. Moon, *Topics on tournaments,* Athena Series, Holt, Rinehart and Winston, New York (1968).

[8] L. Rédei, Ein Kombinatorischer Satz, *Acta Litt. Szeged* **7** (1934), 39–43.

[9] C. Thomassen, Graphs in which every maximal path is contained in a Hamiltonian path, *J. Reine Angew. Math.* **268/269** (1974), 271–282.

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