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Note

Graphs in which each C_4 spans K_4

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Abstract

It is shown that every graph on $n \geqslant 6$ vertices without induced copies of C_4 and K_4-e contains a homogeneous set of size $\lceil \sqrt{n} \rceil$.

1. Introduction

Let Forb(H) denote the family of graphs not containing H as an induced subgraph. Furthermore, by $\alpha(G)$ [$\omega(G)$] we denote the cardinality of a maximum independent set (a maximum complete subgraph) of G and set $\hom(G) = \max\{\alpha(G), \omega(G)\}$. Erdős and Hajnal [2] conjectured that for each H there exists a positive $\varepsilon = \varepsilon(H)$ with the following property: if G has n vertices and $G \in \mathrm{Forb}(H)$ then $\hom(G) \geqslant n^{\varepsilon}$. The conjecture is open even for 'small' H, like C_5 or P_5. It is known that $\varepsilon = \frac{1}{3}$ works for $H = C_4$ (the four cycle) and for $H = K_4 - e$ (the graph on four vertices with five edges), more precisely $\hom(G) \geqslant (2n)^{1/3}$ if G has n vertices and $G \in \mathrm{Forb}(C_4)$ or $G \in \mathrm{Forb}(K_4 - e)$. This was proved in [3] where it was also asked whether $\hom(G) \geqslant \sqrt{n}$ holds if G has n vertices and is $C_4 - forcible$, i.e. $G \in \mathrm{Forb}(C_4) \cap \mathrm{Forb}(K_4 - e)$, so each four cycle induces a K_4 in G. In this note we settle this problem in the affirmative proving the following result.

Theorem. *If G is a C_4-forcible graph on $n \geqslant 6$ vertices then* $\hom(G) \geqslant \lceil \sqrt{n} \rceil$.

Remark. *Clearly, the lower bound $\lceil \sqrt{n} \rceil$ is best possible. Moreover, trivially, the result holds also for $n \leqslant 5$ provided G is not a cycle of length five.*

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2. Elementary properties of C_4-forcible graphs

We start with some simple observations concerning C_4-forcible graphs. Here and below $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and $N(v)$ stands for the neighbourhood of v, i.e. $N(v) = \{w \in V(G) : \{v,w\} \in E(G)\}$. Furthermore, each maximal complete subgraph of a graph we shall call *clique*.

Fact 1. *If K_m is a clique of a C_4-forcible graph G then for every $v \notin V(K_m)$ we have $|N(v) \cap V(K_m)| \leqslant 1$.*

Fact 2. *Let K_ℓ and K_m be cliques of a C_4-forcible graph G. Then one of the following three possibilities holds:*

(i) *K_ℓ and K_m are* intersecting *at a vertex v, i.e. $V(K_\ell) \cap V(K_m) = \{v\}$. In this case there are no edges of G between sets $V(K_\ell) \setminus \{v\}$ and $V(K_m) \setminus \{v\}$;*

(ii) *K_ℓ and K_m are* contiguous *at vertices $v \in V(K_\ell)$ and $v' \in V(K_m)$, i.e. $V(K_\ell) \cap V(K_m) = \emptyset$ and $\{v, v'\}$ is the only edge joining sets $V(K_\ell)$ and $V(K_m)$;*

(iii) *K_ℓ and K_m are* separated, *i.e. $V(K_\ell) \cap V(K_m) = \emptyset$ and no edges of G join sets $V(K_\ell)$ and $V(K_m)$.*

Our next result is slightly more involved. Let us call a sequence of m complete graphs $K_m^{(1)}, K_m^{(2)}, \ldots, K_m^{(m)}$ on m vertices *m-sparse* if
- $|V(K_m^{(i)}) \cap V(K_m^{(j)})| \leqslant 1$ for $1 \leqslant i < j \leqslant m$;
- there is at most one edge between $V(K_m^{(i)})$ and $V(K_m^{(j)})$, for $1 \leqslant i < j \leqslant m$.

Moreover, if $V(K_m^{(i)}) \cap V(K_m^{(j)}) = \{v\}$, then no edges join sets $V(K_m^{(i)}) \setminus \{v\}$ and $V(K_m^{(j)}) \setminus \{v\}$.

Fact 3. *Every m-sparse sequence of complete graphs $K_m^{(1)}, K_m^{(2)}, \ldots, K_m^{(m)}$ contains an independent set $\{v_1, v_2, \ldots, v_m\}$ such that $v_i \in V(K_m^{(i)})$ for $1 \leqslant i \leqslant m$.*

Proof. *By induction.* For $m = 1$ the assertion is obvious. Assume that $m \geqslant 2$ and let v_m be any vertex from $V(K_m^{(m)}) \setminus \bigcup_{i=1}^{m-1} V(K_m^{(i)})$ (since $|V(K_m^{(m)}) \cap V(K_m^{(i)})| \leqslant 1$ for every $1 \leqslant i \leqslant m-1$ such a vertex always exists). Now delete from each $V(K_m^{(i)})$, $1 \leqslant i \leqslant m-1$, the only neighbour of v_m (if such a neighbour does not exist remove any vertex) and use the inductional hypothesis. □

3. Proof of the theorem

We shall show the theorem using the induction with respect to n. For $6 \leqslant n \leqslant 9$ the assertion follows from the fact that $R(3,3) = 6$, whereas for $10 \leqslant n \leqslant 16$ it is an immediate consequence of the equation $R(C_4, K_4) = 10$ (for the values of small off-diagonal Ramsey numbers see [1]). Thus, let G be a C_4-forcible graph on n vertices. From now on we assume that $n = m^2 + 1$, $m \geqslant 4$, and every induced subgraph H of G on at least $(m-1)^2 + 1$ vertices satisfies $\hom(H) = m$. Our goal is to show that $\omega(G) \leqslant m$ implies $\alpha(G) \geqslant m + 1$. Since the inductional step contains many cases, we state each of them as a separate claim.

Fact 4. *Every subgraph of G of at least $(m-1)^2 + m$ vertices contains an independent set of size m.*

Proof. Let H be a subgraph of G on $(m-1)^2 + m$ vertices such that $\alpha(H) < m$. From the assumption $\hom(H) = m$, so H contains a clique $K_m^{(1)}$ on m vertices. Let $v_1 \in V(K_m^{(1)})$ and consider the graph $H_1 = H \setminus \{v_1\}$. Then, $\alpha(H_1) \leqslant \alpha(H) < m$ but, from the assumption, $\hom(H_1) = m$, so H_1 contains a clique $K_m^{(2)}$ on m vertices. Furthermore, due to Fact 2, we have $|V(K_m^{(1)}) \cap V(K_m^{(2)})| \leqslant 1$ and if $V(K_m^{(1)}) \cap V(K_m^{(2)}) = \{v\}$, then no edges join sets $V(K_m^{(1)}) \setminus \{v\}$ and $V(K_m^{(2)}) \setminus \{v\}$. Pick $v_2 \in V(K_m^{(2)})$, set $H_2 = H_2 \setminus \{v_2\}$, and apply the above argument to find in H_2 another clique $K_m^{(3)}$. Continuing this procedure, one can construct in H an m-sparse sequence of cliques $K_m^{(1)}, K_m^{(2)}, \ldots, K_m^{(m)}$. But Fact 3 states that such an H must contain an independent set of size m which contradicts the assumption that $\alpha(H) < m$. □

Fact 5. *If $\omega(G) \leqslant m - 1$ then $\alpha(G) \geqslant m + 1$.*

Proof. If G contains a vertex v of degree at most $2m - 2$ then the subgraph $G \setminus (\{v\} \cup N(v))$, due to our assumption, contains an independent set S of size m, so $S \cup \{v\}$ gives an independent set of size $m + 1$.

Thus, let v be a vertex of G of degree at least $2m - 1$ and let $K_{k_1}, K_{k_2}, \ldots, K_{k_s}$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_s$, denote maximal complete graphs contained in $N(v)$. Note that all these graphs are vertex disjoint and there are no edges between any two of them. Let us choose r in such a way that $m + 1 \leqslant \sum_{i=1}^{r} k_i \leqslant 2m - 2$. Since $k_r \leqslant \cdots \leqslant k_1 \leqslant m - 2$ such a choice is always possible. Now consider the graph $H = G \setminus (\{v\} \cup \bigcup_{i=1}^{r} V(K_{k_i}))$. H has at least $(m-1)^2 + 1$ vertices, so, due to the assumption, contains an independent set S of size m. Furthermore, the fact that G is C_4-forcible implies that each vertex $v \in S$ has at most one neighbour in $\bigcup_{i=1}^{r} V(K_{k_i})$. Since $|\bigcup_{i=1}^{r} V(K_{k_i})| \geqslant m + 1$ one can find a vertex w in $\bigcup_{i=1}^{r} V(K_{k_i})$ such that the set $S \cup \{w\}$ is independent. □

Fact 6. *If G contains a clique K_m, such that some vertex of K_m has no neighbours outside $V(K_m)$, then $\alpha(G) \geqslant m + 1$.*

Proof. Every vertex $v \in V(K_m)$ with $N(v) \subseteq V(K_m)$ extends the largest independent set of $G \setminus V(K_m)$, which, according to Fact 4, has size at least m. □

Fact 7. *If G contains a clique K_m, such that $\omega(G \setminus V(K_m)) \leqslant m-1$, then $\alpha(G) \geqslant m+1$.*

Proof. Because of Fact 6 we may assume that every vertex of K_m has degree at least m. Note also, that if $w, w' \in N(v) \setminus V(K_m)$ for $v \in V(K_m)$ and w and w' are not adjacent, then one can construct an independent set of size $m+1$ by picking one vertex from $N(v') \setminus V(K_m)$ for each $v' \in V(K_m)$, $v' \neq v$, together with w and w'. Thus, let us suppose that the neighbourhood of each vertex $v \in V(K_m)$ consists of two complete graphs.

Let us choose any vertex $v \in V(K_m)$ and set $H = G \setminus (\{v\} \cup N(v))$. H has at least $(m-1)^2 + 1$ vertices, so it contains an independent set S of size m (the assumption rules out cliques on m vertices). Thus, $S \cup \{v\}$ gives the required large independent set. □

Fact 8. *If G contains two separated cliques of size m, then $\alpha(G) \geqslant m+1$.*

Proof. Let K_m and K'_m be two separated cliques of size m in G, and let S be the independent set of size m in $G \setminus V(K_m)$ guaranteed by Fact 4. Note first that every vertex of S sends to $V(K_m)$ at most one edge, so either S can be enlarged by adding to it one vertex from $V(K_m)$ or there are m disjoint edges between S and $V(K_m)$. In particular, we may assume that sets S and $V(K'_m)$ are disjoint. A similar argument shows that either some vertex from $V(K'_m)$ can be used to expand set S or the edges between S and $V(K'_m)$ form a perfect matching. Now, in order to obtain an independent set of size $m+1$, it is enough to remove any vertex s from S and replace it by its neighbours in $V(K_m)$ and $V(K'_m)$. □

Fact 9. *If K_m and K'_m are two contiguous cliques contained in G then either $\alpha(G) \geqslant m+1$ or every vertex $v \in V(K_m)$ has degree m.*

Proof. Let us suppose that $\alpha(G) = m$ and K_m, K'_m are two cliques contiguous at vertices $v \in V(K_m)$ and $v' \in V(K'_m)$. Let S be an independent set of size m in the graph $G \setminus V(K_m)$. Note that, similarly as in the proof of Fact 8, we may and will assume that the edges between S and $V(K_m)$ form a perfect matching.

It is not hard to see that it is enough to study the case when $S \cap V(K'_m) = v'$. Indeed, if $S \cap V(K'_m) = \emptyset$, then, just as in the proof of Fact 8, we may assume that not only the edges between S and $V(K_m)$ but also the edges joining S and $V(K'_m)$ form a perfect matching, and construct an independent set of size $m+1$ by replacing any vertex of S which is adjacent to neither v nor v' by its two neighbours in $V(K_m)$ and $V(K'_m)$. On the other hand, if $S \cap V(K'_m) = w$ then, since w, as an element of S, has neighbour in $V(K_m)$, and K_m and K'_m are contiguous, we must have $w = v'$.

Now let us suppose that $S \cap V(K'_m) = \{v'\}$ and the assertion does not hold, i.e. some vertex of K_m has more than two neighbours outside $V(K_m)$. We consider two cases.

Case 1: *v has a neighbour $w' \neq v'$ outside $V(K_m)$.* Note that, according to Fact 1, $w' \notin V(K'_m)$. Furthermore, w' has no neighbours in $S \setminus \{v'\}$. Indeed, if $u' \in N(w) \cap S \setminus \{v'\}$ and u is the neighbour of u' in K_m then $w'u'uvw'$ is a four cycle and, since G is C_4-forcible, w' has two neighbours, u and v, in K_m, contradicting Fact 1. Thus, the set $S' = S \cup \{w'\} \setminus \{v'\}$ is independent and $S' \cap V(K'_m) = \emptyset$. Now, as we have already observed, we may assume that the edges between S' and $V(K_m)$, as well as the edges between S' and $V(K'_m)$ form a perfect matching, so replacing a vertex of S' adjacent to neither v nor v' by its neighbours in $V(K_m)$ and $V(K'_m)$ leads to an independent set of size $m+1$.

Case 2: *There exists $w \in V(K_m)$, $w \neq v$, with at least two neighbours, say w' and w'', outside $V(K_m)$.* From Fact 2 it follows that neither w' nor w'' belong to $V(K'_m)$. Furthermore, since there is a perfect matching between S and $V(K_m)$, only one of them, say w', belongs to S (if not we can add it to S deleting from S another neighbour of w). Moreover, we may assume that edges between sets $S \setminus \{v'\}$ and $V(K'_m) \setminus \{v'\}$ form a perfect matching, since otherwise there exists $v'' \in V(K'_m)$ with no neighbours in $S \setminus \{v'\}$ and set $S \cup \{v, v''\} \setminus \{v'\}$ is independent. Thus, w' has a neighbour $u \in V(K_m)$. All we have said above remains true also for the independent set $S \cup \{w''\} \setminus \{w'\}$, so w'' must be also adjacent to u. But then vertices w, w', w'' and u lie on the cycle of length four, so u must be adjacent to w which contradicts Fact 2. □

Fact 10. *If no two cliques of size m contained in G have a vertex in common then $\alpha(G) = m+1$.*

Proof. If there is either one clique of size m, or there are no such cliques at all, then the assertion follows from Facts 5 and 7. Thus, let $K_m^{(1)}, K_m^{(2)}, \ldots, K_m^{(\ell)}$, $2 \leqslant \ell \leqslant m$, be the list of all cliques of size m in G. If $K_m^{(1)}$ and $K_m^{(2)}$ are separated, Fact 8 ensures the existence of a large independent set. Thus, suppose that $K_m^{(1)}$ and $K_m^{(2)}$ are contiguous at vertices v_1 and v_2. Fact 9 implies that v_2 is the only neighbour of v_1 outside $V(K_m^{(1)})$. Now let $H = G \setminus (V(K_m^{(1)}) \cup \{v_2, v_3, \ldots, v_\ell\})$, where $v_i \in V(K_m^{(i)})$ for $3 \leqslant i \leqslant \ell$. Then, H has at least $(m-1)^2+1$ vertices, so $\hom(H) = m$, and, since we destroyed all cliques of size m, H contains an independent set S of size m. Hence, $S \cup \{v_1\}$ is an independent set of size $m+1$. □

To conclude the proof of the Theorem it is enough to check that, indeed, the above Facts cover all possible cases and so imply that $\alpha(G) \geqslant m+1$. If either $\omega(G) \leqslant m-1$, or all cliques of size m are vertex disjoint, the existence of a large independent set in G follows from Facts 5 or 10. Suppose now that G contains two intersecting K_m and K'_m. Then, since $m \geqslant 4$, at least one vertex of K_m has at least three neighbours outside $V(K_m)$. Thus, due to Fact 9, we may assume that no clique of size m is contiguous to K_m, so either $\omega(G \setminus V(K_m)) \leqslant m - 1$ and the existence of a large independence set in G follows from Fact 7, or there exists a clique K_m'' separated from K_m and the fact that $\alpha(G) \geqslant m + 1$ is implied by Fact 8. This completes the proof of the Theorem.

References

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