Monochromatic Path Covers

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Abstract

Problems are presented about covering the vertices of edge-colored complete graphs by the vertices of monochromatic paths.

The questions mentioned here originated from path cover problems of ([4], [5]) and from later works with Paul Erdös and L. Pyber ([1], [2]). Perhaps the numbers in parentheses after the problems can be called "Erdös numbers of the second kind", expressing the reward from Paul in US dollars for the answer. (The Erdös number of first kind is simply the Erdös number, on which a special talk was presented during the conference.) There are variations when cycles or trees play the role of paths ([2], [4]), and the infinite versions are also interesting ([6], [7]). Curiously (or naturally), in the infinite case, sharp or almost sharp results are known, but much less is known about the finite case.

Coloring means edge coloring, paths are simple, a monochromatic path is a path whose edges have the same color. The color of a one-vertex path can be arbitrary.

The infinite

The following nice proof of R. Rado is not well known, perhaps because he formulated his theorem in a very general form in [7]. A set of t colors for an r-colored countable complete graph K is perfect if there exists t vertex-disjoint finite paths $P_1 = \dots x_1, \dots, P_t = \dots x_t$, with the following property: P_i is monochromatic in color i and there is an infinite set H of vertices of K such that the edge $x_i y$ is of color i for each i and for all $y \in H$. Select a perfect set of t colors so that t is as large as possible ($t \leq r$). The reader may check that this implies that the vertices of K can be partitioned into t monochromatic finite or one-way infinite monochromatic paths, each of a different color.

The finite

The finite analogue of Rado's result for two colors comes as an easy exercize (it was a footnote in [3], several variants are discussed in [4]). But three colors already present serious problems. The two questions below are about 3-colored (finite!) complete graphs (of at least three vertices).

Problem 1 (25-50)

Is it possible to cover the vertex set of a 3-colored complete graph by the vertices of three (vertex-disjoint) monochromatic paths (of different color)?

In fact, this problem represents four questions compressed into one, depending on inclusion or exclusion of the conditions in parentheses. Needless to say, three can be replaced by r in all of these questions (asked in [5], in more general form in [2]). It is not trivial to show that the minimum number of (vertex-disjoint) monochromatic paths needed to cover the vertices of an r-colored complete graph depends only on r. This (in more general form for cycles) has been shown in [2]. In this direction, the following question had been asked in [2] (again, in slightly stronger form, for cycles).

Problem 2 (25)

Assume that the edges of a balanced complete bipartite graph ($K_{n,n}$) are colored with three colors. Is it possible to cover the vertex set by the vertices of no more than 1995 vertex-disjoint monochromatic paths?

In the case of two colors, the answer is affirmative; in fact 3 works instead of 1995, and it is rather unlikely that a positive proof for problem 2 will require such a big constant. The general problem is to show that $f(r)$ vertex-disjoint monopaths cover the vertices of an r-colored balanced complete bipartite graph.

Since using three colors may seem too complicated, here is another problem just for two colors. In a forthcoming paper [1], it is shown that in any red-blue complete graph, t paths of the same color can cover at least a $\frac{t+1}{t+2}$ fraction of the vertex set. (For $t = 1$ this gives the Ramsey number of the path in the diagonal case). As a simple consequence, the vertices of any red-blue K_n can be covered by at most $2\sqrt{n}$ monopaths of the same color. Perhaps there is a sharper form of this statement, if correct, about the best possible result.

Problem 3 (25)

Is it possible to cover the vertex set of K_n by the vertices of $\sqrt{n}$ monopaths of the same color?

The finite is different!

A month after the Boca Raton meeting, Kathy Heinrich found a nice construction [8], which shows that Rado's result is not true for finite complete graphs colored with $r \geq 3$ colors. Her example for $r = 3$ gives a negative answer to the subproblem of Problem 1 when both conditions (vertex-disjoint, and of different color) are required. On the other hand, Heinrich's example (for any r) can be covered by *two* vertex-disjoint monochromatic paths (of the same color) and also by r monochromatic paths of different colors. So the weaker versions of Problem 1 are still open.

References

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