Minimal Non-Neighborhood-Perfect Graphs

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ABSTRACT

Neighborhood-perfect graphs form a subclass of the perfect graphs if the Strong Perfect Graph Conjecture of C. Berge is true. However, they are still not shown to be perfect. Here we propose the characterization of neighborhood-perfect graphs by studying minimal non-neighborhood-perfect graphs (MNNPG). After presenting some properties of MNNPGs, we show that the only MNNPGs with neighborhood independence number one are the 3-sun and $\overline{3K_2}$. Also two further classes of neighborhood-perfect graphs are presented: line-graphs of bipartite graphs and a $\overline{3K_2}$-free cographs. © 1996 John Wiley & Sons, Inc.

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1. INTRODUCTION

Given a graph $G = (V, E)$ and a vertex $v \in V$, we denote by $G(v)$ the subgraph of G induced by v and all its neighbors. A *neighborhood cover* is a set S of vertices such that every edge of G lies in at least one subgraph $G(s)$ for $s \in S$. The *neighborhood cover number* of G, denoted by $\varrho_N(G)$, is the smallest size of a neighborhood cover of G. This notion was introduced in [6]. Two edges of G are said to be *neighborhood-independent* if there is no vertex v of G such that the two edges both lie in $G(v)$. The *neighborhood independence number,* denoted by $\alpha_N(G)$, is the largest number of pairwise neighborhood-independent edges. Clearly $\alpha_N(G) \leq \varrho_N(G)$ holds for every graph G. A graph is called *neighborhood-perfect* if $\alpha_N(G') = \varrho_N(G')$ holds for every induced subgraph G' of G. This notion introduced in [5] is strongly related to C. Berge's perfectness concept. First we summarize some observations and results in [5] showing the nature of this relationship.

A *trampoline of order k* is a graph with $2k$ vertices $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ such that for every $1 \leq i \leq k$, $\{a_i, a_{i+1}\}$ is an edge and b_i has just two neighbors: a_i and $a_{i+1}(a_{k+1} = a_1)$. The trampoline is complete if $a_1, \ldots, a_k$ induces a k-clique. Complete trampolines are simply called here *suns.* A complete trampoline of order k is called a *k-sun* and is denoted by S_k. Let P_k and C_k denote the induced path and cycle on k vertices. Bipartite graphs and also the class of graphs containing no P_4 and no C_4 as induced subgraph are classes of neighborhood-perfect graphs.

A chordal graph (a graph which is C_k-free for every $k \geq 4$) is neighborhood-perfect iff it does not contain an odd trampoline as induced subgraph (Theorem 4 in [5]).

The complement of a graph G is denoted by $\overline{G}$; C_k and $\overline{C_k}$ is usually called a hole and antihole, respectively. Since odd holes and odd antiholes are not neighborhood-perfect, neighborhood-perfect graphs do not contain an odd hole or an odd antihole as induced subgraph. A graph is said to be *perfect* if for every induced subgraph the maximum size of a clique is equal to the chromatic number. The well-known Strong Perfect Graph Conjecture of C. Berge states that a graph is perfect iff it does not contain an odd hole or an odd antihole as induced subgraph. Graphs without odd holes and antiholes as induced subgraph are nowadays called *Berge graphs.* Consequently, neighborhood-perfect graphs are Berge graphs and the Strong Perfect Graph Conjecture would imply that they are also perfect graphs. The question raised in [5] whether neighborhood-perfect graphs are perfect is still open.

It is worth noting that the class of neighborhood-perfect graphs is not contained in known large classes of perfect graphs. Examples might be obtained using the result (Theorem 4.1 in Section 4) that the line graph $L(G)$ of any bipartite graph G is neighborhood-perfect. Let $K_{m,n}$ denote the complete m by n bipartite graph. Then, $L(K_{2,3})$ is not strongly perfect, $L(K_{3,3})$ is not quasi-parity and does not belong to the class BIP* (for the definition of these perfect classes see [1]). Bipartite graphs such that their line graphs are not preperfect or not locally perfect are given in [4] and [7].

Related to the question whether neighborhood-perfect graphs are perfect, we proposed their characterization in terms of minimal forbidden induced subgraphs. We call a graph G *minimal non-neighborhood-perfect graph,* abbreviated MNNPG, if $\alpha_N(G) < \varrho_N(G)$, but every proper induced subgraph G' of G is neighborhood-perfect, i.e., $\alpha_N(G') = \varrho_N(G')$ holds.

For every odd $k \geq 5$, C_k is an MNNPG, since it is non-neighborhood-perfect, and its one-vertex deleted subgraph, P_{k-1}, is neighborhood-perfect, since it is bipartite. It is easy to check that $\overline{C_k}$, is an MNNPG for $k = 7$ and 8. The complement of three independent edges, $\overline{3K_2}$, is called the *octahedron graph.* The octahedron is an MNNPG with $\alpha_N = 1$ and $\varrho_N = 2$. For every $k \geq 9$, $\overline{C_k}$ is non-neighborhood-perfect, however not minimal, since it contains the

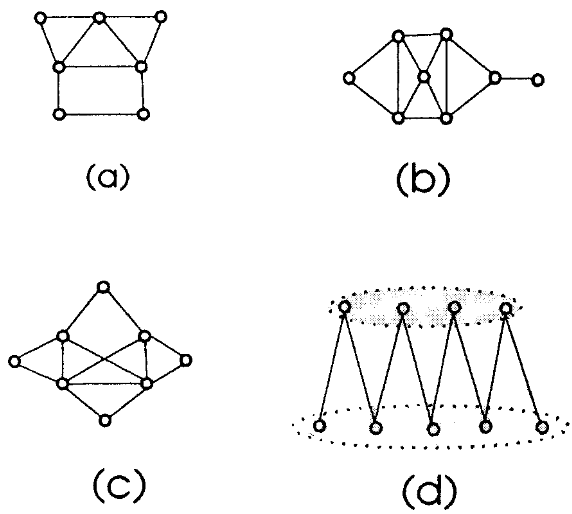

FIGURE 1

octahedron. One may check easily that among non-neighborhood-perfect graphs in [5] odd suns are MNNPGs. The 3-sun is an MNNPG with $\alpha_N = 1$ and $\varrho_N = 2$. In Section 3 we show that the 3-sun and the octahedron are the only MNNPGs with $\alpha_N = 1$ (Theorem 3.5). Observe that the octahedron graph and the 3-sun are examples of non-neighborhood-perfect graphs belonging to large classes of perfect graphs. Indeed, the octahedron is strongly perfect, quasi-parity and locally perfect; the 3-sun is BIP* and preperfect (c.f. [1], [4], and [7]).

In addition to C_5, $\overline{C_7}$, $\overline{C_8}$, and S_5, we give further examples of MNNPGs with $\alpha_N = 2$ and $\varrho_N = 3$ in Figure 1 ((d) is the disjoint union of the cliques K_4 and K_5 together with an alternating 9-path between them).

We know several other MNNPGs with $\alpha_N = 2$, although we are far from their characterization. A challenging open problem in this direction, whether $\varrho_n(G) = \alpha_N(G) + 1$ holds or not for every minimal non-nieghborhood-perfect graph G.

The paper is organized as follows. A reduction operation (edge contraction) preserving the property of being an MNNPG is introduced in Section 2. In Section 3 we show that there are exactly two MNNPGs satisfying $\alpha_N = 1$. Finally, in Sections 4 and 5 two new classes of neighborhood-perfect graphs are presented: the line graphs of bipartite graphs (Theorem 4.1), and the class of P_4- and octahedron-free graphs (Theorem 5.1).

2. CONTRACTION MINIMAL GRAPHS

We present an operation based on edge contraction, which, among others, preserves the property of being an MNNPG. Let G be a graph and let x_1, y_1 be two adjacent vertices of G such that $d_G(x_1) = d_G(y_1) = 2$. Furthermore, let x_1 have another neighbor x and let y_1 have another neighbor y such that x and y are distinct vertices of G. We denote by P the path (x, x_1, y_1, y) of G. In the next two propositions we show how to get a graph G' from G by contracting edges of G such that

$$\alpha_N(G') = \alpha_N(G) - 1 \quad \text{and} \quad \varrho_N(G') = \varrho_N(G) - 1 \qquad (*)$$

are satisfied.

Proposition 2.1. Let G be a graph and let x, y, x_1, y_1 be vertices of G fulfilling the conditions, given above. Furthermore, let $\{x, y\} \in E(G)$. We define G' to be the graph with

$$V(G') = (V(G) - \{x_1, y_1\}) \cup \{z\}, \qquad \text{where } z \notin V(G),$$

and

$$E(G') = (E(G) - \{\{x, x_1\}, \{x_1, y_1\}, \{y_1, y\}\}) \cup \{\{z, x\}, \{z, y\}\}.$$

Then G' satisfies $(*)$.

Proof. Let $M \subseteq E(G)$ be a maximum set of pairwise neighborhood-independent edges of G. If M contains at most one edge of P, then obviously $\alpha_N(G') \geq \alpha_N(G) - 1$. Otherwise, $\{x, x_1\}$ and $\{y_1, y\}$ both belong to M, hence $(M - \{\{x, x_1\}, \{y_1, y\}\}) \cup \{\{x, z\}\}$ is a set of pairwise neighborhood-independent edges of G', again showing $\alpha_N(G') \geq \alpha_N(G) - 1$.

On the other hand, let M' be a maximum set of pairwise neighborhood-independent edges of G'. Suppose, neither $\{x, z\}$ nor $\{y, z\}$ belong to M', then $M' \cup \{\{x_1, y_1\}\}$ is a set of pairwise neighborhood-independent edges of G. Otherwise, w.l.o.g. we may assume $\{x, z\} \in M'$. Then $(M' - \{\{x, z\}\}) \cup \{\{x, x_1\}, \{y_1, y\}\}$ is a set of pairwise neighborhood-independent edges of G, showing that $\alpha_N(G') + 1 \leq \alpha_N(G)$. Consequently, altogether we obtain $\alpha_N(G') = \alpha_N(G) - 1$.

To verify that the neighborhood covering number also decreases by one, let $S \subseteq V(G)$ be a minimum neighborhood cover of G. Clearly, S must contain either x_1 or y_1. W.l.o.g., we may assume $S \cap \{x_1, y_1\} = \{x_1\}$, hence y belongs to S. Then the set $S - \{x_1\}$ is a neighborhood cover of G', showing that $\varrho_N(G') \leq \varrho_N(G) - 1$.

On the other hand, let S' be a minimum neighborhood cover of G', thus $S' \cap \{x, y, z\} \neq \emptyset$. If $z \in S'$ then $(S' - \{z\}) \cup \{x_1, y\}$ is a neighborhood cover of G. Otherwise, we may assume $x \in S'$, thus $S' \cup \{y_1\}$ is a neighborhood cover of G, both showing $\varrho_N(G) \leq \varrho_N(G') + 1$. Altogether, we obtain $\varrho_N(G') = \varrho_N(G) - 1$. ∎

Proposition 2.2. Let G be a graph and let x, y, x_1, y_1 be vertices of G fulfilling the conditions, given above . Furthermore, let $\{x, y\} \notin E(G)$ and there exists no common neighbor $w \in V(G)$ of x and y. We define G' to be the graph with

$$V(G') = (V(G) - \{x_1, y_1\})$$

and

$$E(G') = (E(G) - \{\{x, x_1\}, \{x_1, y_1\}, \{y_1, y\}\}) \cup \{\{x, y\}\}.$$

Then G' satisfies $(*)$.

Proof. Let $M \subseteq E(G)$ be a maximum set of pairwise neighborhood-independent edges of G. If $M \cap (E(G) - (E(G) - E(G')))$ contains at most one edge, then $M \cap E(G')$ is a set of pairwise neighborhood-independent edges of G' of size at least $\alpha_N(G) - 1$. Otherwise, both $\{x, x_1\}$ and $\{y_1, y\}$ belong to M, thus $(M - \{\{x, x_1\}, \{y_1, y\}\}) \cup \{\{x, y\}\}$ is a set of pairwise neighborhood-independent edges of G', also showing $\alpha_N(G') \geq \alpha_N(G) - 1$.

On the other hand, let M' be a maximum set of pairwise neighborhood-independent edges of G'. If $\{x, y\} \in M'$, then $(M' - \{\{x, y\}\}) \cup \{\{x, x_1\}, \{y_1, y\}\}$ is a set of pairwise neighborhood-independent edges of G. Otherwise, $M' \cup \{\{x_1, y_1\}\}$ is a set of pairwise neighborhood-independent edges of G, showing $\alpha_N(G) \geq \alpha_N(G') + 1$. Consequently, $\alpha_N(G') = \alpha_N(G) - 1$.

To verify that the neighborhood covering number also decreases by one, let $S \subseteq V(G)$ be a minimum neighborhood cover of G. W.l.o.g. we may assume $S \cap \{x_1, y_1\} = \{x_1\}$ and $y \in S$. Then, $S - \{x_1\}$ is a neighborhood cover of G', showing $\varrho_N(G') \leq \varrho_N(G) - 1$.

On the other hand, let $S' \subseteq V(G')$ be a minimum neighborhood cover of G'. Since there is no $w \in V(G')$ which is a common neighbor of x and y, it follows that $S' \cap \{x, y\} \neq \emptyset$, say $x \in S'$. Then $S' \cup \{y_1\}$ is a neighborhood cover of G, showing that $\varrho_N(G) \leq \varrho_N(G') + 1$. Thus, $\varrho_N(G) = \varrho_N(G') + 1$ follows. ∎

Before we give the consequence of the propositions to MNNPGs, we mention the case of two adjacent vertices x_1, y_1 of degree 2, which have a common neighbor in G, say x. Then x is cut vertex of G (except when $G = K_3$) and it is not hard to see that $\alpha_N(G - y_1) = \alpha_N(G)$ and $\varrho_N(G - y_1) = \varrho_N(G)$ hold. Consequently, such a graph G cannot be an MNNPG. It is worth noting that a similar easy argument shows that an MNNPG contains no adjacent vertices with the same set of neighbors.

Now, let G be an graph fulfilling the conditions of Proposition 2.1 or Proposition 2.2. It is a matter of routine to check that after the corresponding contraction of G we obtain a graph G' which is also an MNNPG. Therefore, an MNNPG is said to be *contraction minimal* if it cannot be reduced by one of the contractions defined in Proposition 2.1 or Proposition 2.2. An easy consequence is the following

Theorem 2.3. A contraction minimal non-neighborhood-perfect graph different from C_5 has no adjacent vertices of degree two.

This justifies looking for a list of contraction minimal MNNPGs to characterize neighborhood-perfect graphs instead of a list of MNNPGs, i.e., minimal forbidden induced subgraphs. The question whether the "extension", i.e., the inverse of a contraction, of an MNNPG is again an MNNPG remains open so far.

3. MINIMAL NON-NEIGHBORHOOD-PERFECT GRAPHS WITH NEIGHBORHOOD INDEPENDENCE NUMBER ONE

We are going to show that there are exactly two MNNPGs with $\alpha_N(G) = 1$, namely the 3-sun, and $\overline{3K_2}$, the octahedron graph. First, we need some preliminary results. The first lemma is a corollary of a theorem of Cozzens and Kelleher [3].

Proposition 3.1 [3]. If a graph H has at least 2 vertices, is connected and does not contain P_5, C_5 and the complement of the 3-sun as induced subgraph, then H has a dominating edge.

A set $D \subset V(G)$ is called a dominating set of G if every vertex of $V(G) - D$ is adjacent to some element of D. In the case of $D = \{u, v\} \in E(G)$, we say that $\{u, v\}$ is a dominating edge of G_i; if $D = \{x\}$ then x is called a *star vertex* of G. We say that a connected graph has *diameter* d if between any pair of its vertices there is a path containing at most d edges. As usual, K_p denotes the p-clique, a complete graph with p vertices, and qK_p is the union of q disjoint copies of K_p.

Lemma 3.2. If G has diameter 2 and contains neither of C_4, C_5 and the 3-sun as an induced subgraph, then G has a star vertex.

Proof. Since G has diameter 2, for any two non-adjacent vertices u, v of G there is a vertex w adjacent to both, thus no $\{u, v\} \in E(\overline{G})$ can be a dominating edge of the complement $\overline{G}$. Consequently, $\overline{G}$ has no dominating edge. Now, we are going to apply Proposition 3.1 for $H = \overline{G}$. By our first observation and the assumptions on G, we have that H has no dominating edge, furthermore, H does not contain any of $2K_2$, C_5 and the complement of the 3-sun as an induced subgraph. Thus by Proposition 3.1, H cannot be connected. On the other hand, H cannot have two non-trivial connected components since it does not contain $2K_2$. Consequently, H has an isolated vertex, implying that G has a star vertex. ∎

Now, let $G = (V, E)$ be a graph, S and T disjoint subsets of V. We denote by $[S, T]$ the bipartite graph with color classes S, T and edge set $E(S, T) := \{\{s, t\} \in E(G) : s \in S, t \in T\}$. Let $G = (X, Y, E)$ be a bipartite graph. Then $\tilde{G} = (X, Y\{\{x, y\} : x \in X, y \in Y, \{x, y\} \notin E(G)\})$ is called the *bipartite complement* of G.

Lemma 3.3. Let $G = (V, E)$ be a graph, let S and T be disjoint subsets of V such that the bipartite complement of $[S, T]$ has no $3K_2$ as induced subgraph. If T dominates S in the bipartite complement of $[S, T]$, then there are at most two vertices of T dominating S in $\widetilde{[S, T]}$.

Proof. We consider the bipartite graph $\widetilde{[S, T]}$. Let $\tilde{N}(t)$ be the open neighborhood of $t \in T$ in $\widetilde{[S, T]}$. Since T dominates S in $\widetilde{[S, T]}$, we have $\bigcup\{\tilde{N}(t) : t \in T\} = S$. Let $T' = \{t_1, t_2, \ldots, t_k\}$ be a minimal set with the property that $\bigcup\{\tilde{N}(t) : t \in T'\} = S$. We are going to show that $k = |T'| \leq 2$. By the minimality of T', for each $t_i \in T'$ there exists $s_i \in S$ such that $s_i \in \tilde{N}(t_j)$ iff $j = i (1 \leq j \leq k)$. Then $\{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$ induces a kK_2 subgraph of $\widetilde{[S, T]}$, thus, by the assumption, $k \leq 2$ follows. ∎

Before presenting one of the major theorems of this paper, it is worth noting the following relation of our problem to domination.

Proposition 3.4. The following two properties are equivalent:

(i) G satisfies $\alpha_N(G) = 1$ and $\varrho_N(G) > 1$;
(ii) $\overline{G}$ has no isolated vertex and no dominating set inducing $2K_2$, P_4, or C_4.

Proof. Clearly, $\varrho_N(G) > 1$ implies that G has no star vertex, thus $\overline{G}$ has no isolated vertex. If $\overline{G}$ has a dominating set $D = \{a_1, a_2, b_1, b_2\}$ inducing $2K_2$, P_4, or C_4 with $\{a_1, a_2\} \notin E(\overline{G})$ and $\{b_1, b_2\} \notin E(\overline{G})$, then $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are neighborhood-independent edges of G, contradicting $\alpha_N(G) = 1$. On the other hand, two neighborhood-independent edges $\{a_1, a_2\}$ and $\{b_1, b_2\}$ of G induce a $2K_2$, P_4, or C_4 in G and $\overline{G}$, respectively, and there is no vertex adjacent to all the four endpoints of the two edges, consequently, $\{a_1, a_2, b_1, b_2\}$ is a dominating set in $\overline{G}$. ∎

Now, we are going to show that there are exactly two MNNPGs with neighborhood independence number one.

Theorem 3.5. If G is a minimal non-neighborhood-perfect graph and $\alpha_N(G) = 1$, then G is the 3-sun or the octahedron graph.

Proof. Let G be an MNNPG with $\alpha_N(G) = 1$ different from the 3-sun and from the octahedron. Then G cannot have an induced subgraph G' which is an MNNPG with $\alpha_N(G') > 1$. In particular, G is a graph without C_5 and $\overline{C_7}$. Clearly, since G is an MNNPG, it is also a graph containing neither a 3-sun nor an octahedron.

For $x \in V(G)$, we denote by $N(x)$ the (open) neighborhood of x, i.e., the set of all vertices of G adjacent to x, and for a set $A \subset V(G)$, G_A denotes the subgraph of G induced by A. Let $x \in V(G)$ be a vertex of maximum degree in G and select a vertex $y \in V(G) - N(x)$ different from x. Since $\varrho_N(G) > 1$, G has no star vertex and such a vertex y exists. Now we set $A := N(x) \cap N(y)$. A cannot be the empty set. Otherwise, an arbitrary edge incident to x and an arbitrary edge incident to y would be neighborhood-independent, contradicting $\alpha_N(G) = 1$. (Note that every MNNPG is connected.)

Claim 1. G_A has diameter at most two.

If $u, v \in V(G_A)$ and $\{u, v\} \notin E(G)$, then $\alpha_N(G) = 1$ implies that the edges $\{x, u\}$ and $\{y, v\}$ lie in one subgraph $G(w)$ and such a vertex w is necessarily a vertex of A. ∎

Now G_A does not contain a C_5 or a 3-sun as induced subgraph, by the choice of G. Furthermore, G_A does not contain a $\overline{2K_2}$ as induced subgraph, since otherwise these vertices together with x and y would induce an octahedron in G. Thus, Claim 1 and Lemma 3.2 imply that G_A has a star vertex.

Let S be the set of all star vertices in G_A, clearly, S is a clique in G. (If G_A is a single vertex, then $S = A$.) We set $T := N(x) - A$. T is not empty, since otherwise a star vertex of G_A would have degree larger than x, contradicting the choice of x. Furthermore, no vertex $s \in S$ is adjacent to all vertices of T, for otherwise such a vertex would have larger degree in G than x, contradicting the choice of x. Thus, T dominates S in the bipartite complement of $[S, T]$.

Claim 2. If the bipartite graph $[S, T]$ contains a $2K_2$ as induced subgraph, then the corresponding vertices $s_1, s_2 \in S$ and $t_1, t_2 \in T$ induce a C_4 in G.

Since S is a clique, $\{s_1, s_2\} \in E(G)$ holds. If $\{t_1, t_2\} \notin E(G)$, then the vertices s_1, s_2, t_1, t_2, x, y would induce a 3-sun in G. Consequently, we have $\{t_1, t_2\} \in E(G)$, thus $\{s_1, s_2, t_1, t_2\}$ induces a C_4 in G. ∎

Claim 3. Let the vertices $s_1, s_2 \in S$ and $t_1, t_2 \in T$ induce a C_4 in $[S, T]$ such that $\{s_i, t_i\} \notin E(G)$ for $i = 1, 2$. Then $w \in N(y) - N(x)$ implies $\{w, t_1\} \notin E(G)$ or $\{w, t_2\} \notin E(G)$.

Assume $\{w, t_1\} \in E(G)$ and $\{w, t_2\} \in E(G)$. If $\{w, s_i\} \notin E(G)$ for some $i \in \{1, 2\}$ then $\{x, y, w, s_i, t_i\}$ induces a C_5. Otherwise, $\{w, s_1\} \in E(G)$ and $\{w, s_2\} \in E(G)$ implies that $\{x, w, s_1, s_2, t_1, t_2\}$ induces an octahedron. Both contradict the choice of G. ∎

Claim 4. There are at most two vertices $t_1, t_2 \in T$, dominating S in the bipartite complement of $[S, T]$.

Suppose not. T dominates S in the bipartite complement of $[S, T]$. Thus by Lemma 3.3, $\widetilde{[S, T]}$ has a $3K_2$ as induced subgraph. The fact that S is a clique and Claim 2 imply that the vertices of such a $3K_2$ in $\widetilde{[S, T]}$ induce a $\overline{3K_2}$ in G, contradicting the choice of G. ∎

Claim 5. $A - S \neq \emptyset$.

Assume on the contrary that $A = S$. According to Claim 4 we consider two cases.

Case 1. There are two vertices $t_1, t_2 \in T$ dominating S in the bipartite complement of $[S, T]$ and every vertex of T has at least one neighbor in S.

With the notation of the proof of Lemma 3.3, we have that $\tilde{N}(t_1)$ and $\tilde{N}(t_2)$ cover the set S, but neither of the two sets is equal to S itself. We select $s_1 \in \tilde{N}(t_1) - \tilde{N}(t_2)$ and $s_2 \in \tilde{N}(t_2) - \tilde{N}(t_1)$. Consequently, $\{s_1, s_2, t_1, t_2\}$ induces a $2K_2$ in $[S, T]$. Hence, $\{s_1, s_2, t_1, t_2\}$ induces a C_4 in G, by Claim 2, furthermore, $\{s_1, t_1\} \notin E(G)$ and $\{s_2, t_2\} \notin E(G)$ hold. The edges $\{t_1, t_2\}$ and $\{s_1, y\}$ have to lie in some $G(w)$. By Claim 3, w has to be adjacent to x, thus $w \in A$ holds. On the other hand, every vertex of $A = S$ is not adjacent to t_1 or t_2. Consequently, w cannot be in A, thus the two edges are neighborhood-independent, contradicting the choice of G.

Case 2. There is one vertex $t \in T$ which is not adjacent to all the vertices of S.

Let s be an arbitrary vertex of S. The edges $\{t, x\}$ and $\{s, y\}$ have to lie in some $G(w)$. Such a vertex w has to be an element of A, but no vertex of $A = S$ is adjacent to t. Consequently, w cannot be in A, thus the two edges are neighborhood-independent, contradicting the choice of G. ▮

Claim 6. The diameter of G_{A-S} is two.

By Claim 4, at most two vertices of T dominate S in the bipartite complement of $[S, T]$.

Case 1. There are two vertices $t_1, t_2 \in T$ dominating S in the bipartite complement of $[S, T]$ and every vertex of T has at least one neighbor in S.

Define s_1, s_2 as in Case 1 of the proof of Claim 5. We set $B := \{v \in A - S : \{v, t_1\} \in E(G), \{v, t_2\} \in E(G)\}$. Consider the edges $\{t_1, t_2\}$ and $\{y, s_1\}$. $\alpha_N(G) = 1$ implies, that there is a vertex w such that both edges lie in $G(w)$. $w \notin N(x)$ is impossible, since Claim 3 would imply $\{w, t_1\} \notin E(G)$ or $\{w, t_2\} \notin E(G)$. Hence, we have $w \in A$ and by the choice of t_1 and t_2 no vertex of S is adjacent to both of them. Thus, t_1 and t_2 have a common neighbor in $A - S$, i.e., $B \neq \emptyset$. Furthermore, B is a clique, since $\{b_1, b_2\} \notin E(G)$ with b_1, $b_2 \in B$ would result in an octahedron induced by $\{b_1, b_2, s_1, s_2, t_1, t_2\}$.

Since $A - S$ is non-empty by Claim 5, and by the definition of S the graph G_{A-S} is not a clique, it is enough to show that for every pair u, v of non-adjacent vertices in $A - S$ there is a common neighbor $w \in A - S$. Since B is a clique we remain with two subcases.

Subcase 1.1. $u \in B$ and $v \notin B$.

The edges $\{t_1, t_2\}$ and $\{v, y\}$ must lie in some subgraph $G(w)$. By Claim 3, $\{w, x\} \notin E(G)$ is impossible. Hence, w is an element of B, i.e., w is a common neighbor of u and v in $B \subseteq A - S$.

Subcase 1.2. $u \notin B$ and $v \notin B$.

Consider the edges $\{t_1, t_2\}$ and $\{y, u\}$. Now $\alpha_N(G) = 1$ implies that there is a vertex w_1 such that both edges lie in $G(w_1)$. Similar to subcase 1.1, w_1 has to be an element of B and we have $\{w_1, u\} \in E(G)$. Analogously, the neighborhood-independence of the edges $\{t_1, t_2\}$ and $\{y, v\}$ implies the existence of a vertex $w_2 \in B$ such that $\{w_2, v\} \in E(G)$ holds. Since B is a clique, $\{w_1, w_2\} \in E(G)$ holds. If $\{w_1, v\} \in E(G)$ or $\{w_2, u\} \in E(G)$ would hold, then we had a common neighbor of u and v in $A - S$. Thus, let us assume that this is not the case. Consequently, $\{u, v, w_1, w_2\}$ induces a P_4 in G.

The condition u, $v \notin B$ implies that both are not adjacent to at least one of the vertices t_1 and t_2. If $\{u, t_i\} \notin E(G)$ and $\{v, t_i\} \notin E(G)$ for $i \in \{1, 2\}$ then the vertices u, v, w_1, w_2, t_i, and s_j, with $j \in \{1, 2\}$ and $j \neq i$, induce a 3-sun in G. Finally, by symmetry, we remain with $\{u, t_1\} \notin E(G)$ and $\{v, t_2\} \notin E(G)$, but $\{u, t_2\} \in E(G)$ and $\{v, t_1\} \in E(G)$. Now, $\{u, v, y, t_1, t_2\}$ induces a C_5 in G.

Since, all of these consequences contradict the choice of G, the diameter of G_{A-S} is two under the assumptions of case 1.

Case 2. There is one vertex $t \in T$ which is non-adjacent to all the vertices of S.

We set $B := \{v \in A - S : \{v, t\} \in E(G)\}$. Then B cannot be empty, since the edges $\{t, x\}$ and $\{y, s\}$, for some $s \in S$, have to lie in one $G(w)$, and such a vertex w has to be an element of

B. Furthermore, every vertex $v \in A - (S \cup B)$ has at least one neighbor $z \in B$. To see this, consider the edges $\{t, x\}$ and $\{y, v\}$. A vertex z with $G(z)$ containing both edges is a neighbor of v in B. Now, we have to show that every pair of non-adjacent vertices u, $v \in A - S$ has a common neighbor in $A - S$.

Subcase 2.1. $u \in B$ and $v \in B$.

The edges $\{t, u\}$ and $\{y, v\}$ require the existence of a vertex w such that the subgraph $G(w)$ contains both edges. Suppose, $w \notin A$. Hence, $\{w, x\} \notin E(G)$ and $\{w, x, s, t, u, v\}$ induces an octahedron in G, contradicting the choice of G. Therefore, w belongs to B, since no vertex of S is adjacent to the vertex t, and w is a common neighbor of u and v.

Subcase 2.2. $u \in B$ and $v \notin B$.

The edges $\{t, u\}$ and $\{y, v\}$ require the existence of a vertex w such that the subgraph $G(w)$ contains both edges. If $w \in N(x)$, then $w \in B$, and we have a common neighbor of u and v in $B \subseteq A - S$. Thus, we may assume that $w \in N(y) - N(x)$. Hence, $\{s, w\} \in E(G)$ for any $s \in S$. Otherwise, $\{w, t, x, s, y\}$ would induce a C_5 in G.

As mentioned above, v has a neighbor $z \in B$. We may assume $\{z, u\} \notin E(G)$, for otherwise z is already a common neighbor of u and v in $A - S$. If $\{z, w\} \in E(G)$, then $\{w, x, u, z, s, t\}$ induces an octahedron in G. If $\{z, w\} \notin E(G)$, then $\{w, x, y, t, v, u, z\}$ induces a $\overline{C_7}$. Either contradicts the minimality of G. Consequently, u and v always have a common neighbor in $B \subseteq A - S$, if $u \in B$ and $v \notin B$ hold.

Subcase 2.3. $u \notin B$ and $v \notin B$.

Let $w_1 \in B$ be a neighbor of $u \in A - (S \cup B)$. We may assume $\{w_1, v\} \notin E(G)$, otherwise w_1 is already the common neighbor we want to find. By Subcase 2.2, v and w_1 have a common neighbor in B, say $w_2 \in B$. w_2 is a common neighbor of u and v, if $\{w_2, u\} \in E(G)$. On the other hand, $\{w_2, u\} \notin E(G)$ is impossible, since $\{s, t, u, v, w_1, w_2\}$ would induce a 3-sun. ∎

Now, G_{A-S} has diameter two, has neither a C_5 nor a 3-sun, and does not have a $\overline{2K_2}$, since its vertices together with x and y would induce an octahedron in G. Therefore, Lemma 3.2 implies that G_{A-S} has a star vertex, which would be a star vertex of G_A, too. This contradicts the definition of S, and completes the proof of the theorem. ∎

4. NEIGHBORHOOD-PERFECT LINE-GRAPHS

The *line graph* of a graph G, denoted by $L(G)$, is defined on $E(G)$ as its vertex set, and for $e, f \in E(G)$, $\{e, f\}$ is an edge of $L(G)$ iff $e \cap f \neq \emptyset$. In this section we prove that the line graphs of bipartite graphs (which form a well-known large class of perfect graphs) are contained in the class of neighborhood-perfect graphs.

Theorem 4.1. The line-graph of any bipartite graph is neighborhood-perfect.

Proof. Let B be a bipartite graph and $G = L(B)$. We will show that G is neighborhood-perfect by induction on the number of its vertices. The fact is trivial if G has one vertex. Now assume that G has at least two vertices and that the theorem holds for any proper induced subgraph of G. Since ϱ_N and α_N are additive functions over the collection of all components of a graph, we may assume that B is connected. For every vertex x of B we let C_x denote the clique of G consisting of all the edges incident to x in B. Remark that every edge e of G is induced in exactly one such set, which we denote $C(e)$; moreover e is induced in the neighborhood graph $G(y)$, for some $y \in V(G)$, iff $y \in C(e)$. Also observe that $C_x \subseteq C_y$ if and only if x is a vertex of degree one in B.

To calculate $\varrho_N(G)$ and $\alpha_N(G)$, we distinguish between two cases.

Case 1. B has no vertex of degree 1.

Note that under this assumption every clique $C_x(x \in V(B))$ has cardinality at least two. Let us consider a neighborhood cover S of G. By the preceding remark this means that S must intersect every clique C_x of G. In B this translates to the fact that S is a set of edges incident to every vertex, i.e., S is an *edge-cover* of B. It follows that $\varrho_N(G)$ is equal to the minimum size of an edge-cover of B, denoted by $\varrho(B)$.

Now let M be a set of neighborhood-independent edges of G. We define a set X_M of vertices of B by the property that $x \in X_M$ iff some element of M lies in C_x. It is a routine matter to check that X_M is a stable set of B and $|X_M| = |M|$. Conversely, let X be any stable set of B. By the assumption of this case, for each $x \in X$ the clique C_x has cardinality at least two, and we can arbitrarily choose an edge e_x induced by C_x in G. It is easy to check that $\{e_x | x \in X\}$ is a set of cardinality $|X|$ of neighborhood-independent edges of G. It follows that $\alpha_N(G)$ is equal to the maximum size of an independent set of vertices of B, denoted $\alpha(B)$. Then, by a famous theorem of König, $\varrho(B) = \alpha(B)$, so $\varrho_N(G) = \varrho(B) = \alpha_N(G)$ follows.

Case 2. B has a vertex of degree 1.

Let x be a vertex of degree one in B and y its neighbor. Let $e = \{x, y\} \in E(B)$ (e is a vertex of G). Since B has at least two edges and is connected, we may assume that y has other neighbors than x.

Subcase 2.1. y has exactly one neighbor in $B - x$.

Let z be the neighbor of y in $B - x$, and let $f = \{y, z\} \in E(B)$. If B has no further vertex the proof is trivial. So let $x_1, \ldots, x_k (k \geq 1)$ be the neighbors of z in $B - y$. Consider the bipartite graph B' obtained from B by removing x, y, z and adding new vertices $z_1, \ldots, z_k$ and k new edges $z_1x_1, \ldots, z_kx_k$. Notice that B' has two less edges than B. By the induction hypothesis the line-graph $L(B')$ is neighborhood-perfect and it possesses a set M of neighborhood-independent edges and a neighborhood cover S with $|M| = |S|$. Now it is a routine matter to check that $M \cup \{e, f\}$ is a set of neighborhood-independent edges and that $S \cup \{f\}$ is a neighborhood cover of G. Hence $\varrho_N(G) = \varrho_N(L(B')) + 1 = \alpha_N(L(B')) + 1 = \alpha_N(G)$.

Subcase 2.2. y has several neighbors in $B - x$.

Consider the graph $G - e$. By the induction hypothesis this graph possesses a set M of neighborhood-independent edges and a neighborhood cover S with $|M| = |S|$. By the assumption of this subcase, the clique $C_y - \{e\}$ in $G - e$ has cardinality at least two, and so it induces at least one edge. By a remark above this entails that $S \cap (C_y - \{e\}) \neq \emptyset$. Notice that any vertex in $S \cap (C_y - \{e\})$ will also cover the edges incident to e in G, because $e \in C_y$. Consequently S is a neighborhood cover of G. Moreover M is a set of neighborhood-independent edges of G. Hence $\varrho_N(G) \leq |S| = |M| \leq \alpha_N(G)$ and equality follows. ∎

Corollary 4.2. The line-graph of any bipartite multigraph is neighborhood-perfect.

Proof. Let B be a connected bipartite multigraph, G its line-graph and B^* the simple bipartite graph underlying B. If F is a set of parallel edges of B (i.e., edges having the same two extremities) then F induces a clique and a homogeneous set in G. On the basis of this observation is it is a routine task to check that $\varrho_N(G) = \varrho_N(L(B^*))$ and $\alpha_N(G) = \alpha_N(L(B^*))$, except if B^* consists of one single edge (in which case G is a clique, hence trivially neighborhood-perfect). Now the desired conclusion follows from the preceeding theorem. ∎

The preceding result cannot be extended to the class of all line-graphs—consider $L(C_5)$—even if perfection is added as a condition: the line-graph of K_4 is perfect but not neighborhood-perfect (being isomorphic to the octahedron graph). Hence not all $K_{1,3}$-free perfect graphs are neighborhood-perfect. Similarly, line-graphs of simple bipartite graphs are diamond-free, but the preceding theorem cannot be extended to the class of all diamond-free perfect graphs. For example, consider the graph consisting of a cycle on nine vertices $v_0, \dots, v_8$ with chords v_0v_3, v_3v_6, v_6v_0 (since it is contractible to the 3-sun, this graph is not neighborhood-perfect).

5. NEIGHBORHOOD-PERFECT COGRAPHS

Here we characterize neighborhood-perfect graphs in the class of P_4-free graphs, also called *cographs* [2]. In the proof of the next theorem, we will use the following fact due to Seinsche([8]): For any P_4-free graph G having at least two vertices, either G or $\overline{G}$ is disconnected.

Theorem 5.1. A cograph is neighborhood-perfect if and only if it does not contain $\overline{3K_2}$ as an induced subgraph.

Proof. Obviously, if G is neighborhood-perfect it cannot contain the octahedron graph, $\overline{3K_2}$, because it is a (minimal) non-neighborhood-perfect graph. We prove the converse part of the theorem by induction on the order n of the cograph G. The proof is trivial for $n \leq 4$. Now suppose that $n \geq 5$ and that the result holds for every cograph with at most four vertices. If G is disconnected then it suffices to use induction, and the fact that ϱ_N and α_N are additive functions over the collection of all components of a graph, to obtain $\varrho_N(G) = \alpha_N(G)$. So we may now assume that G is connected. It follows from Seinsche's result that $\overline{G}$ is disconnected, with components $\overline{G}_1, \dots, \overline{G}_p (p \geq 2)$, such that each G_i either has just one vertex or is disconnected.

Suppose $p \geq 3$. It must be that at least one of the G_i's has just one vertex, for otherwise we could take a pair of non-adjacent vertices in each of G_1, G_2, G_3 and find that these six vertices induce an octahedron in G. So assume G_3 has just one vertex x. Since x is a star vertex of G, $\varrho_N(G) = \alpha_N(G) = 1$ follows.

Suppose $p = 2$. Let A, B be the vertex sets of G_1, G_2 respectively. If either A or B is of cardinality one then G has a star vertex and we can conclude as above. So we may assume that both A and B have several vertices, and hence G_1 and G_2 are both disconnected. Let $A_1, \dots, A_h$ be the vertex sets of the connected components of G_1, and $B_1, \dots, B_k$ those of G_2, with $k \geq h \geq 2$. Observe that every G_A must be C_4-free, for otherwise the vertices of a C_4 in A_i together with one vertex from B_1 and one vertex from B_2 would induce an octahedron. It is not difficult to prove that any connected C_4-free and P_4-free graph has a star vertex; so each G_{A_i} has a star vertex a_i. Pick one vertex b_i arbitrarily in B_i for $i = 1, \dots, h$. Now letting $S = \{a_1, \dots, a_h\}$ and $M = \{a_1, b_1, \dots, a_h b_h\}$, it is a routine matter to check that S is a neighborhood cover and M is a set of neighborhood-independent edges of G. So $\varrho_N(G) = \alpha_N(G) = h$ follows. ∎

Implicit in the proof of the preceding theorem is an algorithm that finds a neighborhood cover and a set of neighborhood-independent edges of the same size in any octohedron-free cograph. Using the cotree decomposition of cographs, which, as proved in [2], can be found in $O(|E(G)|)$ time, it is not difficult to verify that the algorithm can be implemented to run with the same linear-time complexity.

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