VERTEX COVERING WITH MONOCHROMATIC PATHS

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Abstract: The following theorem is proved. If the edges of $K_n$ are colored red or blue then for each $L$, at least $\frac{n(l+1)}{l+2}$ vertices can be covered by the union of $l$ paths, each monochromatic in the same color. This is essentially best possible for fixed $l$, for $l = 1$ it gives the diagonal path-path Ramsey number, and also shows that $2\sqrt{n}$ monochromatic paths of the same color can cover the vertex set.

Ramsey numbers for paths have been determined in [2]. The diagonal case says that if the edges of $K_n$ are colored with two colors, there exists a monochromatic path with at least $\frac{2n}{3}$ vertices. This can be generalized by asking for any positive integer $l$ the maximum number of vertices coverable by $l$ paths, each monochromatic and having the same color. Notice that if the color of the monochromatic paths can vary then two monochromatic paths can cover the vertex set of any

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2-colored $K_n$. Such problems have been investigated in [1], that paper contains many further references. In this note we prove the following.

**Theorem.** If the edges of $K_n$ colored with two colors then for each $l$ there exist $l$ paths, each monochromatic in the same color, such that they cover at least $\frac{n(l+1)}{l+2}$ vertices of $K_n$.

The result is essentially best possible as shown by the coloring of $K_n$ where the edges of a complete $K_p$ are colored red with $p = \lfloor \frac{2n}{3} \rfloor$ and all other edges are blue. Similar example with $p = \lfloor \frac{2n}{3} \rfloor$ shows that vertex disjoint paths can cover much less, namely $\lfloor \frac{2n}{3} \rfloor + l - 1$ vertices. However, it is possible that the theorem is true if the paths required to be edge disjoint.

**Problem 1.** Is the theorem above true for edge disjoint paths?

The proof of the theorem uses the same technique as used in [2]. The key elements are put together in a lemma. To state it more smoothly, a coloring of $K_n$ always refers to edge colorings with red or blue. A cut coloring is a coloring where the endpoints of a maximum monochromatic (say red) path are connected by a red edge. (In this case all edges with precisely one endpoint in the red cycle are blue.)

**Lemma.** Assume that $K_n$ is given with a coloring which is not a cut coloring. Let $A$ be the vertex set of a maximum monochromatic (say red) path and $B$ is a vertex set in $K_n$ such that $A \cap B = \emptyset$ and $|B| < \lfloor \frac{|A|}{2} \rfloor$. Then there exists a monochromatic blue path containing $B$ and $|B| + 2$ vertices of $A$. If $|A|$ is even and $|B| = \lfloor \frac{|A|}{2} \rfloor$ then there exists a monochromatic blue path with $2|B| + 1$ vertices which covers $|B| + 1$ vertices of $A$.

**Proof.** Consider a coloring of $K_n$ and let $x_1, x_2, \ldots, x_t$ be the vertices of a maximum red path with vertex set $A$. If the edge $x_1, x_t$ is red then we have a cut coloring. Thus we may assume that the edge $x_1, x_t$ is blue.

Observe that the choice of $A$ implies the following property

\[(*) \quad \text{if } Y = \{y_1, y_2, y_3\} \subseteq B \text{ and } X = \{x_i, x_{i+1}\} \text{ for some } i (1 \leq i \leq t-1) \]

\[\text{then either } x_i \text{ or } x_{i+1} \text{ sends at least two blue edges to } Y.\]

Set $C = \{x_2, \ldots, x_{t-1}\}$ and let $P_1$ be a maximal blue path alternating between $C$ and $B$ with endpoints in different classes. In the special case, when $t$ is odd and $|B| = \lfloor \frac{t}{2} \rfloor - 1$ then try to include into $P_1$ a blue edge from $B$ to $x_j$ for some even $j$. This can be done otherwise
there is a blue cycle with $t$ points which implies that we have a cut coloring. If $P_1$ does not include all vertices of $B$ then select again a maximum blue path $P_2$ alternating between $C$ and $B$ and vertex disjoint from $P_1$. Assume first that $|B| < \lceil \frac{t}{2} \rceil$. We claim that $P_1 \cup P_2$ covers all vertices of $B$. Indeed, if $y \in B$ is not covered by the union of $P_1$ and $P_2$ then there exist two consecutive vertices in $C$ neither of them covered by $P_1 \cup P_2$. (The only problematic case is when $t$ is odd and $|B| = \lceil \frac{t}{2} \rceil - 1$ and $P_1 \cup P_2$ covers precisely the vertices of $C$ with odd indices. However, this case is avoided by the definition of $P_1$.) Applying property $(\ast)$ with these two vertices and with $\{y, y_1, y_2\}$ where $y_i$ is the endpoint of $P_i$ in $B$, we get an extension of $P_1$ or $P_2$ contradicting their definitions.

Thus the claim is proved and now the blue path $y_1, x_1, x_t, y_2$ joins $P_1$ and $P_2$ into a blue path with the required property.

If $t$ is even and $|B| = \frac{t}{2}$ then apply the claim to cover $B$ with the exception of one vertex $y$ by the two blue paths $P_1$ and $P_2$. Then the blue path $y_1, x_1, y, x_t, y_2$ joins $P_1$ and $P_2$ into a blue path required by the lemma. $\Diamond$

Corollary 1. (Diagonal case of the path-path Ramsey number established in [2].) In a coloring of $K_n$ there is a monochromatic path of at least $\lfloor \frac{2n}{3} \rfloor + 1$ vertices.

Proof. Assume that the maximum monochromatic (say red) path of a colored $K_n$ has $p$ vertices with vertex set $A$. If $n - p < \lceil \frac{p}{2} \rceil$ then $p$ is as large as required. If the coloring is a cut coloring then there is a blue path with $2\lceil \frac{p}{2} \rceil + 1 > p$ contradicting to the choice of $p$. If the coloring is not a cut coloring, the lemma is applied with selecting $B$ such that $A \cap B = \emptyset$ and $|B| = \lceil \frac{p}{2} \rceil - 1$ if $p$ is odd or $|B| = \frac{p}{2}$ if $p$ is even. The lemma says that there exists a monochromatic blue path with $2(\lceil \frac{p}{2} \rceil) > p$ vertices if $p$ is odd or a monochromatic blue path with $2(\frac{p}{2}) + 1 > p$ vertices if $p$ is even. Both cases contradict the definition of $p$. $\Diamond$

Now we are ready to prove the theorem by induction on $l$, launching it from Cor. 1. Assume it is true for some $l \geq 1$. Select a maximum monochromatic (say red) path of a colored $K_n$ with vertex set $A$ and let $B$ be the complement of $A$ with respect to $V(K_n)$. By the corollary, $|B| = m$ is small and the lemma can be applied to find a blue path which intersects $A$ in $m + 1$ vertices. Delete this set $M$ of $m + 1$ vertices and apply induction to the remaining colored complete graph with $n -$
– \( m - 1 \) vertices. The number of vertices covered by \( l \) paths in that subgraph plus \( m + 1 \) is a lower bound for the number of vertices covered by \( l + 1 \) paths in \( K_n \) since \( M \) is covered by a red path and also by a blue path. Thus it has to be shown that

\[
(n - m - 1) \frac{l + 1}{l + 2} + m + 1 \geq \frac{n(l + 2)}{l + 3}.
\]

Replacing the positive term \( 1 - \frac{l+1}{l+2} \) by zero, the stronger inequality is equivalent to \( m \geq \frac{n}{l+3} \). Thus the induction works unless \( m < \frac{n}{l+3} \).

However, in this case \( n - m > \frac{n(l+2)}{l+3} \) so the red path alone covers the required number of vertices.

**Corollary 2.** The vertex set of a colored \( K_n \) can be covered by no more than \( 2\sqrt{n} \) monochromatic paths of the same color.

**Proof.** Apply the theorem with \( l = \lfloor \sqrt{n} \rfloor \) and use just single vertices to cover the vertices uncovered by the paths.

**Problem 2.** Is Cor. 2 true with \( \sqrt{n} \) instead of \( 2\sqrt{n} \)?

**References**
