Mathematica Pannonica
6/1 (1995), 7 – 10

VERTEX COVERING WITH MONOCHROMATIC PATHS

Paul **Erdős**

Mathematical Institute of the Hungarian Academy of Sciences, H–1364 Budapest, P.O. Box 127, Hungary

András **Gyárfás***

Computer and Automation Institute, Hungarian Academy of Sciences, H–1111 Budapest, Kende u. 13–17, Hungary

Dedicated to Prof. Hans Vogler on the occasion of his 60th birthday

Received October 1994

MSC 1991: 05 C 35, 05 C 15, 05 C 38

Keywords: Vertex covering, monochromatic paths, path-path, Ramsey number, cut coloring.

Abstract: The following theorem is proved. If the edges of K_n are colored red or blue then for each L, at least $\frac{n(l+1)}{l+2}$ vertices can be covered by the union of l paths, each monochromatic in the same color. This is essentially best possible for fixed l, for $l = 1$ it gives the diagonal path-path Ramsey number, and also shows that $2\sqrt{n}$ monochromatic paths of the same color can cover the vertex set.

Ramsey numbers for paths have been determined in [2]. The diagonal case says that if the edges of K_n are colored with two colors, there exists a monochromatic path with at least $\frac{2n}{3}$ vertices. This can be generalized by asking for any positive integer l the maximum number of vertices coverable by l paths, each monochromatic and having the same color. Notice that if the color of the monochromatic paths can vary then two monochromatic paths can cover the vertex set of any

*Supported by OTKA grant 2570.

2-colored K_n. Such problems have been investigated in [1], that paper contains many further references. In this note we prove the following.

Theorem. *If the edges of K_n colored with two colors then for each l there exist l paths, each monochromatic in the same color, such that they cover at least $\frac{n(l+1)}{l+2}$ vertices of K_n.*

The result is essentially best possible as shown by the coloring of K_n where the edges of a complete K_p are colored red with $p = \lfloor \frac{n(l+1)}{l+2} \rfloor$ and all other edges are blue. Similar example with $p = \lfloor \frac{2n}{3} \rfloor$ shows that vertex disjoint paths can cover much less, namely $\lfloor \frac{2n}{3} \rfloor + l - 1$ vertices. However, it is possible that the theorem is true if the paths required to be edge disjoint.

Problem 1. Is the theorem above true for edge disjoint paths?

The proof of the theorem uses the same technique as used in [2]. The key elements are put together in a lemma. To state it more smoothly, a coloring of K_n always refers to edge colorings with red or blue. A *cut coloring* is a coloring where the endpoints of a maximum monochromatic (say red) path are connected by a red edge. (In this case all edges with precisely one endpoint in the red cycle are blue.)

Lemma. *Assume that K_n is given with a coloring which is not a cut coloring. Let A be the vertex set of a maximum monochromatic (say red) path and B is a vertex set in K_n such that $A \cap B = \emptyset$ and $|B| < \lceil \frac{|A|}{2} \rceil$. Then there exists a monochromatic blue path containing B and $|B| + 2$ vertices of A. If $|A|$ is even and $|B| = \frac{|A|}{2}$ then there exists a monochromatic blue path with $2|B| + 1$ vertices which covers $|B| + 1$ vertices of A.*

Proof. Consider a coloring of K_n and let $x_1, x_2, \ldots, x_t$ be the vertices of a maximum red path with vertex set A. If the edge x_1, x_t is red then we have a cut coloring. Thus we may assume that the edge x_1, x_t is blue.

Observe that the choice of A implies the following property

(*) *if $Y = \{y_1, y_2, y_3\} \subseteq B$ and $X = \{x_i, x_{i+1}\}$ for some i $(1 \leq i \leq t-1)$ then either x_i or x_{i+1} sends at least two blue edges to Y.*

Set $C = \{x_2, \ldots, x_{t-1}\}$ and let P_1 be a maximal blue path alternating between C and B with endpoints in different classes. In the special case, when t is odd and $|B| = \lceil \frac{t}{2} \rceil - 1$ then try to include into P_1 a blue edge from B to x_j for some even j. This can be done otherwise there is a blue cycle with t points which implies that we have a cut coloring. If P_1 does not include all vertices of B then select again a maximum blue path P_2 alternating between C and B and vertex disjoint from P_1. Assume first that $|B| < \lceil \frac{t}{2} \rceil$. We claim that $P_1 \cup P_2$ covers all vertices of B. Indeed, if $y \in B$ is not covered by the union of P_1 and P_2 then there exist two consecutive vertices in C neither of them covered by $P_1 \cup P_2$. (The only problematic case is when t is odd and $|B| = \lceil \frac{t}{2} \rceil - 1$ and $P_1 \cup P_2$ covers precisely the vertices of C with odd indices. However, this case is avoided by the definition of P_1.) Applying property $(*)$ with these two vertices and with $\{y, y_1, y_2\}$ where y_i is the endpoint of P_i in B, we get an extension of P_1 or P_2 contradicting their definitions.

Thus the claim is proved and now the blue path y_1, x_1, x_t, y_2 joins P_1 and P_2 into a blue path with the required property.

If t is even and $|B| = \frac{t}{2}$ then apply the claim to cover B with the exception of one vertex y by the two blue paths P_1 and P_2. Then the blue path y_1, x_1, y, x_t, y_2 joins P_1 and P_2 into a blue path required by the lemma. $\Diamond$

Corollary 1. (Diagonal case of the path-path Ramsey number established in [2].) *In a coloring of K_n there is a monochromatic path of at least $\lfloor \frac{2n}{3} \rfloor + 1$ vertices.*

Proof. Assume that the maximum monochromatic (say red) path of a colored K_n has p vertices with vertex set A. If $n - p < \lceil \frac{p}{2} \rceil$ then p is as large as required. If the coloring is a cut coloring then there is a blue path with $2\lceil \frac{p}{2} \rceil + 1 > p$ contradicting to the choice of p. If the coloring is not a cut coloring, the lemma is appplied with selecting B such that $A \cap B = \emptyset$ and $|B| = \lceil \frac{p}{2} \rceil - 1$ if p is odd or $|B| = \frac{p}{2}$ if p is even. The lemma says that there exists a monochromatic blue path with $2(\lceil \frac{p}{2} \rceil) > p$ vertices if p is odd or a monochromatic blue path with $2(\frac{p}{2}) + 1 > p$ vertices if p is even. Both cases contradict the definition of p. $\Diamond$

Now we are ready to prove the theorem by induction on l, launching it from Cor. 1. Assume it is true for some $l \geq 1$. Select a maximum monochromatic (say red) path of a colored K_n with vertex set A and let B be the complement of A with respect to $V(K_n)$. By the corollary, $|B| = m$ is small and the lemma can be applied to find a blue path which intersects A in $m+1$ vertices. Delete this set M of $m+1$ vertices and apply induction to the reamining colored complete graph with $n -$

$- m - 1$ vertices. The number of vertices covered by l paths in that subgraph plus $m+1$ is a lower bound for the number of vertices covered by $l+1$ paths in K_n since M is covered by a red path and also by a blue path. Thus it has to be shown that

$$(n-m-1)\frac{l+1}{l+2}+m+1 \geq \frac{n(l+2)}{l+3}$$

Replacing the positive term $1-\frac{l+1}{l+2}$ by zero, the stronger inequality is equivalent to $m \geq \frac{n}{l+3}$. Thus the induction works unless $m < \frac{n}{l+3}$. However, in this case $n-m > \frac{n(l+2)}{l+3}$ so the red path alone covers the required number of vertices. $\Diamond$

Corollary 2. *The vertex set of a colored K_n can be covered by no more than $2\sqrt{n}$ monochromatic paths of the same color.*

Proof. Apply the theorem with $l = \lfloor\sqrt{n}\rfloor$ and use just single vertices to cover the vertices uncovered by the paths. $\Diamond$

Problem 2. Is Cor. 2 true with $\sqrt{n}$ instead of $2\sqrt{n}$?

References

[1] ERDŐS, P.; GYÁRFÁS, A. and PYBER, L.: Vertex coverings by monochromatic cycles and trees, *Journal of Comb. Th.* (B) **51** (1991), 90–95.

[2] GERENCSÉR, L. and GYÁRFÁS, A.: On Ramsey type problems, *Annales Univ. Sci. Eötvös Sect. Math.* **10** (1967), 167–170.