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Linear Sets with Five Distinct Differences among Any Four Elements*

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As a generalization of the concept of Sidon sets, a set of real numbers is called a (4, 5)-set if every four-element subset determines at least five distinct differences. Let $g(n)$ be the largest number such that any n-element (4, 5)-set contains a $g(n)$-element Sidon set (i.e., a subset of $g(n)$ elements with distinct differences). It is shown that $(1/2+\varepsilon)\,n \leqslant g(n) \leqslant 3n/5+1$, where ε is a positive constant. The main result is the lower bound whose proof is based on a Turán-type theorem obtained for sparse 3-uniform hypergraphs associated with (4, 5)-sets. © 1995 Academic Press, Inc.

1. INTRODUCTION

A set $\{r_1, r_2, ..., r_n\}$ of real numbers (or positive integers) is called here a Sidon set (usually called a B_2-sequence) if the sums $r_i + r_j$, $1 \leqslant i \leqslant j \leqslant n$, are all distinct (cf. [HR, ET, and SO]). Equivalently, in a Sidon set there are no equal differences between its elements. Erdős and Sós introduced the following generalization of Sidon sets. A set of real numbers is called (p, q)-set if every p-element subset determines at least q distinct differences. (Note that Sidon sets are equivalent to (4, 6)-sets.) They observed that any (4, 5)-set of n elements contains a cn-element Sidon set and asked about the maximum of c.

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Let $g(n)$ be the largest number such that any (4, 5)-set with n elements contains a Sidon set of $g(n)$ elements. In Section 2 we show that $(n+1)/2 \leqslant g(n) \leqslant 3n/5+1$ (Corollaries 2.2 and 2.5). Our main result is that $g(n) \geqslant (1/2+\varepsilon)\, n$, where ε is a constant (Corollary 3.3). The proof, in Section 3, is based on a Turán-type result pertaining to sparse 3-uniform hypergraphs associated with (4, 5)-sets as follows. Erdős and Sós observed that any (4, 5)-set $\{r_1, r_2, ..., r_n\}$ satisfies the following conditions. If $r_i - r_j = r_l - r_k > 0$, then $r_l = r_j$, i.e., the elements involved in repeated differences from 3-term arithmetic progressions, the midpoints of these 3-term arithmetic progressions are different, and two 3-term arithmetic progressions have at most one common element. Based on these properties one may associate a 3-uniform hypergraph $H = (R, E)$ with a (4, 5)-set $R = \{r_1, r_2, ..., r_n\}$ as follows: R is the vertex set of H, and for $1 \leqslant i < j < k \leqslant n$, $\{r_i, r_j, r_k\}$ belongs to the edge set E of H iff $r_i < r_j < r_k$ form an arithmetic progression. This hypergraph H is called the A.P.-hypergraph of R.

The (4, 5)-property of a set R of n elements transforms into the following hypergraphic properties on its A.P.-hypergraph: it is 3-uniform, it has at most $n-2$ hyperedges, and its distinct hyperedges have at most one common vertex. In addition, we show that the A.P.-hypergraph of any (4, 5)-set has no partial hypergraph H_0 shown in Fig. 1 (Proposition 2.6).

Obviously, Sidon sets contained in (4, 5)-set R correspond to independent sets of its A.P.-hypergraph $H = (R, E)$, thus the independence number, $\alpha(H)$, becomes the size of the largest Sidon set of R. Since the complement of any independent set of H is a transversal for its hyperedges, an upper bound on the transversal number $\tau(H) = |R| - \alpha(H)$ yields a lower bound for $\alpha(H)$. Our bound $g(n) \geqslant (1/2+\varepsilon)\, n$ (see Corollary 3.3) is derived from the second part of the following result.

THEOREM A. *Assume H is a 3-uniform hypergraph with n vertices and with at most n edges. Then, $\tau(H) \leqslant n/2$. Moreover, if no two edges of H intersect in two vertices, and H contains no H_0 shown in Fig. 1, then $\tau(H) \leqslant (\frac{1}{2} - \varepsilon)\, n$ with some positive constant ε.*

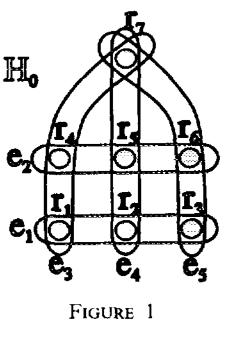

FIGURE 1

It is worth noting that the first part of the theorem gives a Turán-number for "3-graphs" which is already known from Sidorenko's results [Si]. The advantage of the proof given here is that it leads to the characterization of all extremal hypergraphs (i.e., those with $\tau(H) = n/2$). We state this result as Theorem B below (its proof is in a preliminary version of this paper). For $n \geqslant 4$ even, let $J(n)$ be the 3-uniform hypergraph defined on the set $\{0, 1, ..., n-1\}$ with the n triplets $\{i, i+1, i+n/2\}$, $i = 0, 1, ..., n-1$, and for $n \geqslant 3$, let $L(n)$ be the 3-uniform hypergraph defined on $\{0, 1, ..., n-1\}$ with all triplets $\{i, i+1, i+3\}$, $i = 0, 1, ..., n-1$ (mod n arithmetic in both cases).

Theorem B. *Let H be a connected 3-uniform hypergraph with n vertices and at most n edges. Then $\tau(H) \leqslant n/2$, and $\tau(H) = n/2$ if and only if either $H = J(n)$ or $H = L(8)$.*

Note that $L(8)$ (more generally, disjoint copies of it) is extremal and has no edges intersecting in two vertices. Therefore the exclusion of H_0 is crucial in improving the bound from $\frac{1}{2}n$ to $(\frac{1}{2} - \varepsilon)\, n$ in Theorem A. Our proof yields $\varepsilon = (76 \times 141)^{-1}$; on the other hand, $\varepsilon \leqslant 1/26$ is shown by the disjoint union of copies of the hypergraph $L(13)$. Theorem A will be proved in Section 3 (as Theorem 3.2).

2. Properties of (4, 5)-Sets

First a combinatorial property of (4, 5)-sets is formulated in terms of hypergraph colorings. A hypergraph is said to be k-colorable if its vertex set has a partition into k independent classes, i.e., subsets containing no hyperedges.

Proposition 2.1. *The A.P.-hypergraph of any (4, 5)-set is 2-colorable.*

Proof. Let R be a (4, 5)-set with A.P.-hypergraph H. Let us define a graph $G = (R, U)$ representing the hyperedges of H as follows. Any pair $x < y$ of R forms an edge in U if and only if there exists a 3-term arithmetic progression $x < y < z$ in R. By definition, and since the midpoints of 3-term arithmetic progressions are distinct, G is acyclic. Hence G is a forest, and in particular, bipartite. Since every hyperedge of H is represented by the edges in U, H is 2-colorable. ∎

The larger color class of the 2-coloring of an A.P.-hypergraph H in Proposition 2.1 results in an independent set containing more than half of the vertices. This leads to the first lower bound on the maximum size of a Sidon set contained in (4, 5)-sets.

COROLLARY 2.2. $g(n) \geqslant (n+1)/2$.

To get an upper bound on $g(n)$ we consider a particular set constructed on Fibonacci numbers. Let $f(i)$ denote the ith Fibonacci number defined by the recursion: $f(0)=1$, $f(1)=1$, and $f(i)=f(i-1)+f(i-2)$, for $i \geqslant 2$.

PROPOSITION 2.3. *The n Fibonacci numbers* $f(2), f(3), ..., f(n+1)$ *form a* $(4, 5)$*-set.*

Proof. Let i, j, k, l be integers with $2 < i < j \leqslant k < l \leqslant n+1$. First we show that the equality

$$f(j)-f(i)=f(l)-f(k) \tag{1}$$

holds only if $i=j-2$, $k=j$, and $l=j+1$.

Since

$$f(l)-f(k)=f(l-2)+f(l-1)-f(k) \geqslant f(l-2),$$
$$f(l)-f(k)=f(l)-(f(k+1)-f(k-1)) \geqslant f(k-1),$$

and $f(j)-f(i)<f(j)$, Eq. (1) is possible only if $j>\max\{k-1,\ l-2\}$. Hence $k=j$ and $l=j+1$ follows. Then, (1) becomes $f(j)-f(i)=f(j+1)-f(j)=f(j-1)$ which implies $i=j-2$.

Observe that among any four elements of the Fibonacci sequence there exists at most one triple in the form $\{f(j-2), f(j), f(j+1)\}$. Thus, using our first argument, there is at most one equal pair among the six possible pairwise differences. This concludes the proof of the proposition. ∎

PROPOSITION 2.4. *For each* $n \geqslant 4$, *let* $F(n)$ *be the A.P.-hypergraph of the Fibonacci set* $\{r_i = f(i+1) : i=1, ..., n\}$. *Then* $\tau(F(n)) = \lfloor 2(n-1)/5 \rfloor$.

Proof. Let T be a minimum transversal of $F(n)$. Since r_1 and r_2 are vertices of degree one, one may assume that $r_1, r_2 \notin T$. Consequently, at least one of r_3 and r_4 belongs to T. Observe that, for any $i \geqslant 3$, $r_{i+2}, r_{i+3}, r_{i+4} \notin T$ implies $r_i, r_{i+1} \in T$. Assume that there are p (obviously, pairwise disjoint) 5-tuples with this property covering a subset P of $5p$ vertices. Then, for any $j \geqslant 3$, $r_{j+2}, r_{j+3} \notin T \cup P$ implies $r_j, r_{j+1} \in T$. This shows that at least half of the vertices of $\{r_4, r_5, ..., r_n\} \backslash P$ belong to T. Therefore,

$$\begin{aligned}
|T| &= |T \cap P| + |T \cap (\{r_4, r_5, ..., r_n\} \backslash P)| \geqslant 2p + \frac{n-3-5p}{2} \\
&= 2p + \frac{n-1-5p}{2} - 1 \geqslant 2p + \frac{2(n-1-5p)}{5} - 1 \\
&= \frac{2(n-1)}{5} - 1.
\end{aligned}$$

Observe that the bound is sharp only if there is equality at both places of inequality, that is, if $r_3 \notin T \cup P$ and $n-1-5p=0$. The first condition, if true, implies $n-2-5p>0$, thus excluding the second one. Hence, $|T| \geqslant \lfloor 2(n-1)/5 \rfloor$ follows.

The reverse inequality easily follows by constructing the minimum transversal for every $4 \leqslant n_0 \leqslant 8$, then, for any $n=5p+n_0$, by completing the minimum transversal on the lower n_0 vertices with the pairs $\{r_{5i+n_0+1}, r_{5i+n_0+2}\}$, for $i=0, 1, ..., p-1$. ∎

Proposition 2.4 implies that the A.P.-hypergraph of an n-element Fibonacci-set has no independent set larger than $3n/5+1$. This results in the following upper bound on the maximum size of a Sidon-set in a (4, 5)-set.

COROLLARY 2.5. $g(n) \leqslant 3n/5+1$. ∎

Let $H_0=(R, E)$ be the hypergraph defined by $R=\{r_1, ..., r_7\}$ and $E=\{e_1, ..., e_5\}$, where $e_1=\{r_1, r_2, r_3\}$, $e_2=\{r_4, r_5, r_6\}$, $e_3=\{r_1, r_4, r_7\}$, $e_4=\{r_2, r_5, r_7\}$, and $e_5=\{r_3, r_6, r_7\}$ (see Fig. 1).

PROPOSITION 2.6. *The A.P.-hypergraph of any* (4, 5)*-set contains no* H_0 *as a partial hypergraph.*

Proof. Assume indirectly that the vertices of H_0 are real numbers which form a (4, 5)-set $R=\{r_1, ..., r_7\}$ such that each edge of H_0 is an edge in the *A.P.*-hypergraph of R. By definition, R contains no four distinct numbers satisfying $r_i - r_j = r_l - r_k$. We say that r_j is the midpoint of the hyperedge $e_h=\{r_i, r_j, r_k\}$, if $2r_j = r_i + r_k$.

Case a: r_7 is the midpoint of some hyperedge. Assume that r_x and r_y are the two elements of R which are not midpoints $(1 \leqslant x < y \leqslant 6)$. For every hyperedge $e_h=\{r_a, r_b, r_c\}$, $1 \leqslant h \leqslant 5$, let us introduce the form $w(e_h)=2r_c - r_a - r_b$, where r_c is the midpoint of e_h, and let us consider the sum $W=w(e_1)+w(e_2)-w(e_3)-w(e_4)-w(e_5)$. Observe that the coefficient of each term r_i in the expansion of W comes from the forms belonging to the hyperedges incident with r_i. Using this, it is easy to check that the coefficients of r_x, r_y, and r_7 all become zero in W. Actually, we get $W=3r_i+3r_j-3r_k-3r_l$, where r_i, r_j are the midpoints of the hyperedges not containing r_7, and r_k, r_l are the two further midpoints different from r_7. Since $w(e_h)=0$, for every $1 \leqslant h \leqslant 5$, we obtain $W/3=r_i+r_j-r_k-r_l=0$. Hence $r_i-r_k=r_l-r_j$ follows which violates the (4, 5)-property of R.

Case b: r_7 is not a midpoint. By symmetry, one may assume that r_1 is the second vertex which is not a midpoint, and the midpoint of e_1 is r_2.

Then, the midpoints of all hyperedges become uniquely determined. This yields the following system of midpoint equations:

$$(e_1)\ \ 2r_2 = r_1 + r_3, \qquad (e_2)\ \ 2r_6 = r_4 + r_5, \qquad (e_3)\ \ 2r_4 = r_1 + r_7,$$
$$(e_4)\ \ 2r_5 = r_2 + r_7, \qquad (e_5)\ \ 2r_3 = r_6 + r_7.$$

By choosing $r_7 = 0$ and by introducing the parameter $\beta = r_5 - r_3$ we easily get the following solution: $r_4 = 5\beta$, $r_1 = 10\beta$, $r_5 = 3\beta$, $r_3 = 2\beta$, $r_6 = 4\beta$, and $r_2 = 6\beta$. Thus $r_2 - r_4 = \beta = r_5 - r_3$ follows which violates the (4, 5)-property of R.

Both cases led to contradiction. ∎

3. Minimum Transversals in 3-Uniform Hypergraphs with n Vertices and with at Most n Edges

In this section we give bounds on the transversal number of sparse 3-uniform hypergraphs. Let $H = (V, E)$ be a 3-uniform hypergraph, $|V| = n$, $|E| \leqslant n$. A transversal $T = S \cup M$ will be constructed in two steps as follows.

Greedy Step. Start with $S = \emptyset$ and if $S = \{x_1, x_2, ..., x_i\}$ is already defined then let x_{i+1} be a vertex which covers the maximum number of edges uncovered by S. If this maximum is at least three, x_{i+1} is added to S; otherwise the greedy step stops. When leaving the greedy step, let H_2 be the partial hypergraph formed by the hyperedges of H not covered by S.

Pairing Step. Select as many pairs of (distinct) intersecting edges of H_2 as possible. Then M is defined by choosing a vertex from the intersection of each pair of selected edges and by adding one vertex from each of the remaining (not selected) edges. Note that since H_2 has maximum degree at most two, M is a minimum transversal of H_2.

In order to estimate the size of the transversal $S \cup M$ given by this procedure we need the following matching lemma. (As usual, $\nu(G)$ denotes the size of the maximum matching of graph G.)

Lemma 3.1. *Let G be a multigraph with maximum degree three. Then $|E(G)| \leqslant 4\nu(G)$, with equality if and only if each component of G is a single vertex or a triangle with a double edge. Moreover, if G has no multiple edges then $|E(G)| \leqslant 3.5\nu(G)$.*

Proof. Set $\nu = \nu(G)$ and select a maximum matching $Z = \{x_1 y_1, ..., x_\nu y_\nu\}$ such that $|\{i\colon x_i y_i \text{ has a parallel edge in } G,\ 1 \leqslant i \leqslant \nu\}|$ is as large as possible.

Let $G|Z$ be the subgraph of G induced by the vertices of Z. For each i, $1 \leqslant i \leqslant v$, define

$$s_i = \frac{d_{G|Z}(x_i) + d_{G|Z}(y_i)}{2} + d_G(x_i) - d_{G|Z}(x_i) + d_G(y_i) - d_{G|Z}(y_i).$$

The choice of Z ensures $d_G(x_i) - d_{G|Z}(x_i) + d_G(y_i) - d_{G|Z}(y_i) \leqslant 2$, and in case of equality we say that $x_i y_i$ is full. Now it is immediate that $s_i \leqslant 4$ and $s_i = 4$ implies that $x_i y_i$ is full. From the maximality of the matching, G has no edges with both endpoints in $V(G) - V(G|Z)$, thus

$$|E(G)| = \sum_{i=1}^{v} s_i \leqslant 4v$$

which proves the required inequality.

In case of equality, $s_i = 4$; consequently, $x_i y_i$ is full, for every $i = 1, 2, ..., v$. Assume that, for some i, $x_i y_i$ has no parallel edge in G. Then there exists an edge $e \in E(G)$ and $j \neq i$ such that e intersects both $x_i y_i$ and $x_j y_j$. It is easy to check that there is an alternating chain e_1, $x_i y_i$, e, $x_j y_j$, e_2 with endpoints in $V(G) \backslash V(G|Z)$. This contradicts the maximality of Z. Therefore, in case of $|E(G)| = 4v$, $x_i y_i$ has parallel edges for all i, so G is the union of single vertices and triangles with a double edge.

To prove $|E(G)| \leqslant 3.5v$ for simple graphs, in a similar spirit, takes much more effort. Fortunately, this result is a special case of a theorem of Chvátal and Hanson [CH] cited in [LP] as Theorem 3.4.6. ∎

THEOREM 3.2. *Assume that H is a 3-uniform hypergraph with n vertices and with at most n edges. Then*

(a) $\tau(H) \leqslant n/2$.

(b) *If no two edges of H intersect in two vertices and H does not contain H_0 as a partial hypergraph, then, with some positive constant ε, $\tau(H) \leqslant (\frac{1}{2} - \varepsilon) n$.*

Proof. Let $H = (V, E)$ be a 3-uniform hypergraph, $|V| = n$, $|E| \leqslant n$. Let $T = S \cup M$ be the transversal constructed in the greedy and pairing steps defined at the beginning of this section. Assume that at the end of the greedy step $S = \{x_1, x_2, ..., x_k\}$ and $S = S_0 \cup S_1$, where S_0 is the set of those vertices in S which cover at least four uncovered edges when added to S, and $S_1 = S - S_0$ is the set of those vertices in S which cover exactly three uncovered edges when added to S. It is possible that S_0 or S_1 (or both) are empty sets.

The edge set of H is partitioned into $E = E_0 \cup E_1 \cup E_2$, where E_0 is the set of all edges covered by S_0, $E_1 \subseteq E \backslash E_0$ is the set of edges covered by S_1, and $E_2 = E - (E_0 \cup E_1)$ is the set of edges not covered by S. The set $V \backslash S$ is partitioned as $V_0 \cup V_1 \cup V_2$, where V_i is the set of vertices covering exactly i edges of E_2 ($i = 0, 1, 2$). Set $p_i = |V_i|$, for $i = 0, 1$, and 2. It is clear from the definitions that

$$|S| + \sum_{i=0}^{2} p_i = n \tag{1}$$

and

$$3\,|S| + |E_2| \leqslant |E_0 \cup E_1| + |E_2| \leqslant n. \tag{2}$$

Let M be the minimum transversal of the hypergraph $H_2 = (V_1 \cup V_2, E_2)$ obtained in the pairing step and denote by G_2 the dual hypergraph of H_2. Note that G_2 is a multigraph with p_2 edges, and each edge has multiplicity at most two. As a hypergraph, G_2 has also p_1 (possibly multiple) one-element hyperedges. However, by discarding them, G_2 becomes a multigraph with maximum degree three. By the definition of M, $|M| = (|E_2| - 2\nu(G_2)) + \nu(G_2) = |E_2| - \nu(G_2)$. From the first part of Lemma 3.1 we have $\nu(G_2) \geqslant p_2/4$ which implies

$$|M| \leqslant |E_2| - \frac{p_2}{4}. \tag{3}$$

Combining (1), (2), and (3), then using the identity $3\,|E_2| = p_1 + 2p_2$ we easily get

$$\tau(H) \leqslant |S| + |M| \leqslant \frac{n}{2} - \frac{p_0}{4}. \tag{4}$$

This proves part (a) of the theorem. The proof of part (b) is in the same spirit. First (1) and (2) are refined as

$$|S_0| + |S_1| + \sum_{i=0}^{2} p_i = n \tag{1'}$$

and

$$3\,|S_1| + |E_2| \leqslant n. \tag{2'}$$

Due to the first condition in (b), there are no multiple edges in G_2 so the second part of Lemma 3.1 gives

$$|M| \leqslant |E_2| - \frac{p_2}{3.5}. \tag{3'}$$

Combining these inequalities as before, we get

$$\tau(H) \leqslant \frac{n}{2} - \left(\frac{|S_0| + p_0}{4} + \frac{p_2}{28}\right). \tag{4'}$$

Since our aim is to show $\tau(H) \leqslant (\frac{1}{2} - \varepsilon)\, n$, assume (with a suitable chosen $\varepsilon > 0$ adjusted later) that

$$\frac{|S_0| + p_0}{4} + \frac{p_2}{28} \leqslant \varepsilon n. \tag{5}$$

Let A be the set of vertices in V_1 for which there exists an edge $e \in E_2$ such that $e \cap A \neq \emptyset$ and $e \cap V_2 \neq \emptyset$. Since each $x \in V_2$ is in two edges of E_2, it follows that

$$|A| \leqslant 4p_2. \tag{6}$$

Set $B = V_1 \backslash A$ and $F = \{e \in E_2 \colon e \subseteq B\}$. The definition of B and V_1 implies that F is a set of pairwise disjoint hyperedges; moreover, $\bigcup \{f \colon f \in F\} = B$. Let us define the hypergraph $H_3 = (V_3, E_3)$, where $V_3 = S_1 \cup B$ and $E_3 = \{e \in E \colon e \subseteq V_3\}$. We shall prove that

$$\tau(H_3) \leqslant (\tfrac{1}{2} - \tfrac{1}{76})\, n. \tag{7}$$

Before proving (7), we show how to finish the proof of the theorem. Inequality (5) implies $|S_0| + p_0 + p_2 \leqslant 28\varepsilon n$ and thus $p_2 \leqslant 28\varepsilon n$. From these, together with (6), $|S_0| + p_0 + p_2 + |A| \leqslant 140\varepsilon n$ follows. Since $V_3 = V \backslash (S_0 \cup V_0 \cup V_2 \cup A)$, and by using (7), we get

$$\begin{aligned} \tau(H) &\leqslant \tau(H_3) + |V \backslash V_3| = \tau(H_3) + (|S_0| + |V_0| + |V_2| + |A|) \\ &\leqslant (\tfrac{1}{2} - \tfrac{1}{76})\, n + 140\varepsilon n. \end{aligned}$$

Then the proof concludes by choosing ε to satisfy $\varepsilon = 1/76 - 140\varepsilon$, which gives $\varepsilon = (141 \times 76)^{-1}$.

Proof of (7). Note that S_1 and one vertex from each $e \in F$ is a transversal of H_3 which gives the trivial estimate

$$\tau(H_3) \leqslant |S_1| + |F|. \tag{8}$$

Case 1. $|S_1| \leqslant (\frac{1}{4} - \delta)\, n$. Using this, together with (8), yields

$$\begin{aligned} \tau(H_3) &\leqslant |S_1| + |F| \leqslant |S_1| + \frac{n - |S_1|}{3} = \frac{2\,|S_1|}{3} + \frac{n}{3} \\ &\leqslant \frac{2}{3}\left(\frac{1}{4} - \delta\right) n + \frac{n}{3} = \left(\frac{1}{2} - \delta\right) n \end{aligned}$$

Case 2. $|S_1| \geqslant (\frac{1}{4}-\delta)\, n$. The trivial estimate (8) is improved by the following procedure. The degree of a vertex $x \in V_3$, $d_{H_3}(x)$ is simply referred to as $d(x)$ in this part of the proof. Let $E(x)$ denote the set of edges of H_3 containing x. An *improving block* $B_i \subset V_3$ will be defined for a pair (x_i, A_i) such that $x_i \in S_1$ and A_i is a set of $d(x_i)$ edges of F with the following property: there exists a set $T_i \subseteq B$ of $d(x_i)$ vertices which covers both the edges of A_i and the edges of $E(x_i)$. For such a pair (x_i, A_i) let $B_i = \{x_i\} \cup (\bigcup \{f : f \in A_i\})$. Note that by the definition of S_1, $|S_1 \cap e| \leqslant 1$, for every edge e of H_3. Thus the maximum degree of H_3 is at most three, hence the improving blocks may have 1, 4, 7, and 10 vertices.

We wish to select pairwise disjoint improving blocks by a greedy procedure. Assume that B_i is already selected, for $i = 1, 2, ..., m$. Set $U = B \cap (B_1 \cup B_2 \cup \cdots \cup B_m)$. Since the maximum degree of H_3 is at most three, $|U| \leqslant 9m$ and there are at most $18m$ vertices $x \in S_1$ for which $(\bigcup \{e : e \in E(x)\}) \cap U \neq \emptyset$. Therefore, if $|S_1| > 18m$, there exists $x_{m+1} \in S_1$ for which $(\bigcup \{e : e \in E(x_{m+1})\}) \cap U = \emptyset$. Then a new improving block B_{m+1}, disjoint from $B_1, ..., B_m$, is defined as follows.

Assume that $d(x_{m+1}) = d$ $(d \leqslant 3)$ and let $E(x_{m+1}) = \{e_1, ..., e_d\}$. If $d = 0$ then set $B_{m+1} = \{x_{m+1}\}$ and $T_{m+1} = \emptyset$. If $d = 1$, then set $B_{m+1} = \{x_{m+1}\} \cup f_i$, where $f_i \in F$ and $f_i \cap e_1 \neq \emptyset$; furthermore, define $T_{m+1} = f_i \cap e_1$. If $d = 2$, one may clearly select distinct hyperedges $f_i, f_j \in F$ such that $f_i \cap e_1 \neq \emptyset$ and $f_j \cap e_2 \neq \emptyset$. Then the next improving block is $B_{m+1} = \{x_{m+1}\} \cup (f_i \cup f_j)$ with $T_{m+1} = (f_i \cap e_1) \cup (f_j \cap e_2)$. The last case, $d = 3$, is the critical part of the proof. Since H_0 is forbidden in H, in particular in H_3, there exist distinct hyperedges $f_i, f_j, f_k \in F$ such that $f_i \cap e_1$, $f_j \cap e_2$, and $f_k \cap e_3$ are nonempty. Now the next improving block is $B_{m+1} = \{x_{m+1}\} \cup (f_i \cup f_j \cup f_k)$ with $T_{m+1} = (f_i \cap e_1) \cup (f_j \cap e_2) \cup (f_k \cap e_3)$.

The argument above shows that there are at least $|S_1|/18$ pairwise disjoint improving blocks. Each improving block gains one over the trivial estimate (8) since in each of them T_i can be selected as part of a transversal. Therefore,

$$\tau(H_3) \leqslant |S_1| + |F| - \frac{|S_1|}{18}. \tag{9}$$

Observe that $|S_1| + 3\,|F| \leqslant n$ (since H has n vertices) and $3\,|S_1| + |F| \leqslant n$ (since H has no more than n edges). These inequalities imply $|S_1| + |F| \leqslant n/2$. Using this, together with (9), and since we are in Case 2, $\tau(H_3)$ can be estimated follows.

$$\tau(H_3) \leqslant |S_1| + |F| - \frac{|S_1|}{18} \leqslant \frac{n}{2} - \frac{|S_1|}{18} \leqslant \left(\frac{1}{2} - \frac{1}{18}\left(\frac{1}{4} - \delta\right)\right) n.$$

Comparing the estimates obtained in Cases 1 and 2 for $\tau(H_3)$, $\delta = \frac{1}{76}$ is selected as the solution of $\delta = \frac{1}{18}(\frac{1}{4} - \delta)$. This proves (7) and concludes the proof of the theorem. ∎

Since the A.P.-hypergraph of any (4, 5)-set satisfies the conditions of Theorem 3.2, we immediately have the follwing corollary.

COROLLARY 3.3. *Every* (4, 5)*-set of n elements contains a Sidon set with at least* $(1/2 + \varepsilon)\, n$ *elements.*

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