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Independent Transversals in Sparse Partite Hypergraphs

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An $[n,k,r]$-hypergraph is a hypergraph $\mathscr{H}=(V,E)$ whose vertex set V is partitioned into n k-element sets $V_1, V_2, \ldots, V_n$ and for which, for each choice of r indices, $1 \le i_1 < i_2 < \ldots < i_r \le n$, there is exactly one edge $e \in E$ such that $|e \cap V_i| = 1$ if $i \in \{i_1, i_2, \ldots, i_r\}$ and otherwise $|e \cap V_i| = 0$. An independent transversal of an $[n,k,r]$-hypergraph is a set $T = \{a_1, a_2, \ldots, a_n\} \subseteq V$ such that $a_i \in V_i$ for $i = 1, 2, \ldots, n$ and $e \not\subseteq T$ for all $e \in E$. The purpose of this note is to estimate $f_r(k)$, defined as the largest n for which any $[n,k,r]$-hypergraph has an independent transversal. The sharpest results are for $r = 2$ and for the case when k is small compared to r.

A *sparse partite hypergraph* is defined as a hypergraph $\mathscr{H}=(V,E)$ whose vertex set V is partitioned into n k-element sets $V_1, V_2, \ldots, V_n$ and for which, for each choice of r indices, $1 \le i_1 < i_2 < \ldots < i_r \le n$, there is exactly one edge $e \in E$ such that $|e \cap V_i| = 1$ if $i \in \{i_1, i_2, \ldots, i_r\}$ otherwise $|e \cap V_i| = 0$. Notice that a sparse partite hypergraph has kn vertices and $\binom{n}{r}$ edges and if $k = 1$, it is a complete r-uniform hypergraph on n vertices. If $k \ge 2$ and $r < n$, there are many non-isomorphic sparse partite hypergraphs and we shall use the term $[n,k,r]$-hypergraph for any of them. It is worth mentioning that the $[n,k,r]$-hypergraph whose vertex set is the union of pairwise disjoint edges seems to be an important one (in this case $k = \binom{n-1}{r-1}$). That hypergraph is used, for example, by Nešetřil and Rödl [3] in their construction of hypergraphs with large girth and large chromatic number.

Let $\mathscr{H}=(V,E)$ be an $[n,k,r]$-hypergraph with vertex partition $V = V_1 \cup V_2 \cup \ldots \cup V_n$, $|V_i| = k$ for $i = 1, 2, \ldots, n$. An *independent transversal* of $\mathscr{H}$ is a set $T = \{a_1, a_2, \ldots, a_n\} \subseteq V$ such that $a_i \in V_i$ for $i = 1, 2, \ldots, n$ and $e \not\subseteq T$ for all $e \in E$.

The purpose of this note is to estimate $f_r(k)$, defined as the largest n for which any $[n,k,r]$-hypergraph has an independent transversal. To avoid trivial cases, it is always assumed that $2 \le r < n$, $k \ge 2$. The graph theoretical problem of estimating $f_2(k)$ arose in

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connection with structural properties of special graphs but seems to have an independent interest. This is determined within the constant factor $2e$. It is easy to see that $f_2(2) = 3$, but more work is needed to establish $f_2(3) = 7$, $f_3(2) = 5$. As in the case of Ramsey numbers, exact values of $f_r(k)$ seem to be difficult to find, but the bounds are much better here. It seems interesting that the reasonably close lower and upper bounds are *both* found using the probabilistic method. The upper bound (Proposition 1) comes from the 'basic' method and the lower bound (Propositions 2 and 3) comes from the Local Lemma. From these bounds, asymptotics of $f_r(k)$ are given if k is small compared to r (Propositions 4 and 5). However, if r is fixed, the probabilistic upper bound has an extra $\log k$ factor. We could get rid of this only when $r = 2$ with a constructive example (Proposition 6). This, together with the lower bound of Theorem 2, gives the following:

$$(1 + o(1))(2e)^{-1}k^2 < f_2(k) < (1 + o(1))k^2,$$

which might be a sign that $f_2(k)$ can be estimated more accurately. Indeed, Raphael Yuster has reported [4] that it is possible to replace $2e$ by 2.1 using a constructive method. The authors also appreciate his valuable remarks leading to the revised version of this note.

Finally, let us mention that the probabilistic method has been applied to somewhat similar problems by Alon and Spencer [1, Chapter 5, Proposition 5.3].

Proposition 1. *If*

$$k^n(1 - 1/k^r)^{\binom{n}{r}} < 1, \tag{1}$$

then $f_r(k) < n$.

Proof. The probability space is of all $[n, k, r]$-hypergraphs on a fixed vertex set with equal probability. Let A_T be the event that a transversal T is independent. Clearly,

$$\mathrm{Prob}(A_T) = \left(1 - \frac{1}{k^r}\right)^{\binom{n}{r}}$$

since an r-element subset of T is an edge with probability $1/k^r$. There are k^n transversals so condition (1) ensures the existence of an $[n, k, r]$-hypergraph with no independent transversals. □

Proposition 2. *If*

$$e\left(\binom{n}{r} - \binom{n-r}{r}\right)\frac{1}{k^r} < 1, \tag{2}$$

then $f_r(k) \geq n$.

Proof. For a fixed $[n, k, r]$-hypergraph $\mathscr{H}$, with a vertex partition $V = V_1 \cup V_2 \cup \ldots \cup V_n$, $|V_i| = k$ for $i = 1, 2, \ldots, r$, let the probability space consist of all transversals of $\mathscr{H}$, where each transversal is equally likely. For each edge e of $\mathscr{H}$, let A_e be the event that a transversal of $\mathscr{H}$ contains all vertices of e. Furthermore, define

$$V_e = \bigcup_{V_i \cap e \neq \emptyset} V_i \, .$$

Notice that the event A_e is independent of the set of events

$$\{A_{e'} : V_e \cap V_{e'} = \emptyset\} \, .$$

Thus, the dependency graph of the events A_e has maximum degree bounded from above by $\binom{n}{r} - \binom{n-r}{r} - 1$. Hence, by the Local Lemma [2, 1], if (2) holds, $Prob(\bigcap_e \bar{A}_e) > 0$ and, consequently, $\mathscr{H}$ contains an independent transversal. $\square$

Since

$$\binom{n}{r} - \binom{n-r}{r} \leq r\binom{n-1}{r-1} = \frac{r^2}{n}\binom{n}{r},$$

we immediately obtain the following result from Proposition 2.

Proposition 3. *If*

$$er^2\binom{n}{r} < nk^r, \tag{3}$$

then $f_r(k) \geq n$.

It turns out that the bounds given by Propositions 1 and 3 are quite tight in the case when r is large and k is not much larger than r. Here and below $o_t(u)$ denotes a quantity such that $o_t(u)/u \to 0$ as $t \to \infty$.

Theorem 1. *Let $r \to \infty$ and $k \leq \exp\exp(o_r(r))$. Furthermore, let $n_{k,r}$ be the largest natural number n for which (3) holds. Then $f_r(k) = (1 + o_r(1))n_{k,r}$.*

Proof. Let $k \leq \exp\exp(r/\omega(r))$ for some function $\omega(r)$ that tends to infinity with r. We may (and shall) assume that $\omega(r) \leq \log r$. Thus, in order to verify the assertion, it is enough to check whether (1) holds for $n = (1 + 1/\sqrt{\omega(r)})n_{k,r}$. But for such $n_{k,r}$ and r large enough we have

$$\begin{aligned}
k^n(1-1/k^r)^{\binom{(1+1/\sqrt{\omega(r)})n_{k,r}}{r}} &\leq \exp\left(n\log k - \frac{1}{k^r}\binom{(1+1/\sqrt{\omega(r)})n_{k,r}}{r}\right)\\
&\leq \exp\left(n\log k - \left(1+\frac{1}{\sqrt{\omega(r)}}\right)^r \frac{1}{k^r}\binom{n_{k,r}}{r}\right)\\
&= \exp\left(n\log k - \left(1+1/\sqrt{\omega(r)}\right)^r (n/r^2)(1/e - o_r(1))\right)\\
&\leq \exp(n(\exp(r/\omega(r)) - \exp(r/(2\sqrt{\omega(r)})))) < 1.
\end{aligned}$$

$\square$

One can easily estimate the value of $n_{k,r}$ using Stirling's formula. Thus, from Theorem 1, we immediately get the following two consequences.

Proposition 4. *Let $k \geq 2$ be a fixed natural number and $r \to \infty$. Furthermore, let $a > 0$ denote the solution of the equation $a^a/(a-1)^{a-1} = k$. Then $f_r(k) = (1 + o_r(1))ar$.*

Proposition 5. *Let $k(r)$ be a function that tends to infinity as $r \to \infty$ in such a way that $k \leq \exp\exp(o_r(r))$. Then $f_r(k) = (1 + o_r(1))rk/e$.*

Unfortunately, when k is much larger than r, and, in particular, when r is fixed and k increases, our estimate of $f_r(k)$ is not as accurate anymore.

Theorem 2. *If $r \leq o_k(\sqrt{k})$,*

$$(1 + o_k(1))((r-1)!/(er))^{\frac{1}{r-1}} k^{\frac{r}{r-1}} < f_r(k) < (1 + o_k(1))(r!)^{\frac{1}{r-1}} k^{\frac{r}{r-1}} (\ln k)^{\frac{1}{r-1}} .$$

Proof. Since $r = o_k(\sqrt{k})$, we can approximate $\binom{n}{r}$ in (1) and (3) by $(1 + o_k(1))n^r/r!$, and the assertion easily follows. □

When $r = 2$ we can get rid of the $(\ln k)^{1/(r-1)}$ factor.

Proposition 6. *If an affine plane of order $k + 1$ exists, $f_2(k) < (k + 1)^2$.*

Proof. Let A_{k+1} be an affine plane of order $k + 1$ with points $\{1, 2, \ldots, (k + 1)^2\}$, and let C_1, C_2, …, C_{k+2} be the parallel classes of A_{k+1}. In order to define a $[(k + 1)^2, k, 2]$-graph G, view the vertices of G as the $k \times (k+1)^2$ matrix $V = [v_{ij}]$, the columns of which are the partite classes of G. For $i = 1, 2, \ldots, k$, two vertices v_{ij} and v_{il} of the ith row are adjacent if and only if j and l are in the same block in C_i. This makes each row of V the union of $k + 1$ copies of K_{k+1}. Add to G further edges (*e.g.* the pairs covered by C_{k+1} and C_{k+2}) arbitrarily to get a $[(k+1)^2, k, 2]$-graph. To prove the proposition, we have to show that G has no independent transversals. Indeed, only $k + 1$ vertices can be chosen from each row to an independent transversal and this gives at most $k(k + 1) < (k + 1)^2$ elements. (Note that our argument shows that in the assertion $(k + 1)^2$ can be changed to $k^2 + k + 1$ by truncation of A_{k+1}.) □

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