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On a Ramsey Type Problem of Shelah

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Shelah used a Ramsey type statement in his new proof of Van der Waerden's theorem ([5]). It was formulated in [3] as a coloring problem on mesh clique graphs: find the smallest $n = F(r)$ such that in any r-coloring of the edges of $K_n \times K_n$ there exists a $K_2 \times K_2$ whose opposite edges have the same color. Here the trivial upper bound on $F(r)$ is improved using the Fisher inequality on hypergraphs with restricted intersections.

A fragment of Shelah's proof of the Van der Waerden theorem [5] is connected to a Ramsey type problem. This problem have been formulated in [3] as follows. Assume that n is a positive integer and define the *mesh clique graph* G_n with vertex set

$$\{(i,j): \quad 1 \le i \le n,\, 1 \le j \le n,\;\; i,j \text{ are integers } \} \tag{1}$$

and two distinct vertices of G_n are adjacent if and only if they agree in their first or in their second coordinate. Vertices of G_n with fixed first (second) coordinates are called the columns (rows) of G_n, and these terms are also used for subgraphs induced by rows or columns. The graph G_n is clearly isomorphic to $K_n \times K_n$. Assume that i, j, k, l are integers satisfying $1 \le i < j \le n$, $1 \le k < l \le n$. The four-cycle in G_n induced by the vertices (i,k), (i,l), (j,k), (j,l) is called here a *rectangle* (referring to the position of its points in the planar grid). An r-coloring of G_n means edge coloring with numbers $1, 2, \ldots, r$. A rectangle of G_n is called here *alternatig* with respect to a given r-coloring if its opposite edges have the same color (a monochromatic rectangle is a special case). Define $F(r)$ as the smallest n for which under any r-coloring of G_n there exists an alternating rectangle

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(a.r.). Colorings of G_n without a.r.-s are called good colorings. Clearly, $F(r)-1$ is the largest integer n for which there exists a good coloring of G_n In [3] (p.68) one can read: "a polynomial upper bound on $F(r)$ might well lead to a towerian upper bound to $HJ(2,t)$ [the Hales Jewett number for two colors and t symbols]. Even if not, it is a certainly interesting problem for its own sake."

Notice that the problem has its natural off-diagonal extension. A pair $[m,n]$ might be called r-minimal if under any r-coloring of $G_{m,n}$ there is an a.r. but this is not true if m or n is decreased. ($G_{m,n}$ is defined by changing the range of j to $1 \le j \le m$ in (1)). The problem is to find or bound the r-minimal pairs. These pairs are clearly symmetric, $[m,n]$ is r-minimal iff $[n,m]$ is r-minimal, so $m \le n$ can be stipulated in the definition. Under this assumption $F(r)$ is the smallest n for which $[n,n]$ is r-minimal.

The trivial bounds on $F(r)$ are

$$r+1 \le F(r) \le r^{\binom{r+1}{2}}+1 \tag{2}$$

The lower bound in (2) follows simply by coloring all columns of G_r with a different color (in fact, this shows that if $[m,n]$ is r-minimal then $r+1 \le m$). Heinrich proved that $F(r) \ge c \cdot r^3$ with positive c and mentioned that $F(r) > r^3$ can be proved for prime r with her method ([4]). Faudree, Szőnyi and the author used projective spaces to obtain good colorings and have shown that $F(r) > r^3$ for prime power r ([2]). These lower bounds together with the upper bound in (2) imply $F(2) = 9$. The method in [2] would provide $r^{t+1} < F(r)$ if the t-dimensional projective space of order r is "complementary", which means that there exists a permutation on the points which maps each hyperplane to the complement of some hyperplane. Blokhuis observed that for a complementary projective space $t \le r$ so the the best one can hope from the method is $r^{r+1} < F(r)$. Notice that the existence of complementary t-dimensional projective spaces of order r such that t tends to infinity with r would kill the possibility of a polynomial $F(r)$ and the attempt to squeeze out a towerian upper bound from Shelah's proof this way. However, no complementary spaces are known to the author for $t \ge 3$. Any example would improve the lower bound on $F(r)$.

The upper bound in (2) had been used in Shelah's proof. It follows from the pigeonhole principle. Set $n = \binom{r+1}{2}+1$ and consider any r-coloring of the edges of $G_{r+1,n}$. Two columns of this graph must be colored exactly the same way and one can select two horizontal edges of the same color between these columns to find the a.r. In fact, the argument shows (together with the lower bound) that $[r+1, r^{\binom{r+1}{2}}+1]$ is an r-minimal pair.

In this paper this simple pigeonhole argument is extended to improve the upper bound in (2). The improvement is $r^{\binom{r-1}{2}+1}$ which sounds great in other problems but here of course is not enough to decrease the exponent of the trivial bound. The proof uses the uniform Fisher inequality on intersecting set systems and probably further improvements are possible using the results and techniques on set systems with restricted intersections. A survey is in the preliminary version of the book of Babai and Frankl [1]. However, to lower the exponent $\binom{r+1}{2}$ in (2) probably further ideas are needed.

Theorem. *If $r \geq 3$ then $F(r) < r^{\binom{r+1}{2}} - r^{\binom{r-1}{2}+1} + 1$.*

Proof. Set $n = r^{\binom{r+1}{2}} - r^{\binom{r-1}{2}+1} + 1$. Assume that an r-coloring is given on $G_{m,n}$. We show that either there exists an a.r. or $m \leq r^{2 \cdot r-1} + r - 1$ (note that m is smaller than n for $r \geq 3$). Thus the proof gives a stronger off-diagonal result.

Represent the coloring of the columns of $G_{m,n}$ by an $\binom{m}{2} \times n$ matrix M whose rows are indexed by unordered pairs of R, $|R| = m$ and whose columns are indexed by elements of $C = \{1, 2, ..., n\}$. The element $M[ij, k]$ $(1 \leq i < j \leq m, 1 \leq k \leq n)$ is the color of the edge ij in the k-th column of $G_{m,n}$.

Each row ij of M gives a partition $P_{ij} = [A^1_{ij}, A^2_{ij} \dots A^r_{ij}]$ on C by defining

$$A^k_{ij} = \{p \in C : M[ij, p] = k\} \text{ for } k = 1, 2, \dots r.$$

The proof is based on intersection properties of the P_{ij}-s, the sets A^k_{ij} are referred as *blocks*. A *j-section* is the intersection of j non-parallel blocks. A set of non-parallel blocks (or partitions) is *well-covered* if the union of their row indices have at most $r+1$ points in R. Define for convenience

$$n_j = r^{\binom{r+1}{2} - j}$$

Now the pigeonhole argument is generalized in the following claim.

Claim 1. *Assume that $I \subseteq C$ is a j-section of well-covered blocks. Then $|I| \leq n_j$.*

Proof. Assume that the graph $R_1 \subset \binom{R}{2}$ defines the row indices of the j blocks in question, $|R_1| = j$. Since the blocks are well-covered there exists $R_2 \subset \binom{R}{2}$ such that $R_1 \cap R_2 = \oslash$ and $R_1 \cup R_2 = K_{r+1} \subset \binom{R}{2}$. Consider the submatrix $M' \subset M$ whose rows are indexed by K_{r+1} and whose columns are indexed by I. The rows of M' indexed by R_1 are constant thus M' has at most $r^{|R_2|} = n_j$ different columns. If $|I| > n_j$ then two columns of M' are the same and the a.r. is present by the same argument used in the trivial upper bound. ∎

Claim 2. *Any* $\binom{r-1}{2}$ *partitions have an* $\binom{r-1}{2}$*-section* I *with* $|I| \geq n_{\binom{r-1}{2}} - r + 1$.

Proof. The $r^{\binom{r-1}{2}}$ $\binom{r-1}{2}$-sections of the $\binom{r-1}{2}$ partitions cover C. If all of them have cardinality less than required then

$$n = |C| \leq r^{\binom{r-1}{2}} \cdot (n_{\binom{r-1}{2}} - r) = r^{\binom{r+1}{2}} - r^{\binom{r-1}{2}+1}$$

contradicting to the definition of n. ∎

Apply Claim 2 with $\mathcal{P} = \{P_{ij} : 1 \leq i < j \leq r-1\}$ and let I be the $\binom{r-1}{2}$-section guaranteed.

Claim 3. *Assume that* j *satisfies* $r \leq j \leq m$. *Then* P_{1j} *has a block* B *such that* $|I \cap B| = n_{\binom{r-1}{2}+1}$.

Proof. Since I is covered by the blocks of P_{1j},

$$|I| \leq \sum_{k=1}^{r} |A_{1j}^k \cap I|$$

and all terms are at most $n_{\binom{r-1}{2}+1}$ by Claim 1 since the set $\mathcal{P} \cup \{P_{1j}\}$ is well-covered. Thus denying the claim means that all terms are at most $n_{\binom{r-1}{2}+1} - 1$. This implies $|I| \leq n_{\binom{r-1}{2}} - r$, contradicting the definition of I. ∎

Choose a block of P_{1j} for each j, $j = r, r+1, \ldots, m$ according to Claim 3. W.l.o.g. these blocks are A_{1j}^1. Set $B_j = I \cap A_{1j}^1$. From Claim 3. we know that $|B_j| = n_{\binom{r-1}{2}+1}$.

Claim 4. $|B_u \cap B_v| = n_{\binom{r-1}{2}+2}$ *for all* u,v *satisfying* $r \leq u < v \leq m$.

Proof. The key observation is that $\mathcal{P} \cup \{P_{1u}, P_{1v}\}$ is still well-covered, since its row indices form a K_{r-1} plus two edges incident to the same vertex of this complete graph. Thus each term in

$$|I \cap B_u| = \sum_{k=1}^{r} |I \cap B_u \cap A_{1v}^k|$$

is at most $n_{\binom{r-1}{2}+2}$. But from Claim 3 $|I \cap B_u| = n_{\binom{r-1}{2}+1}$ and this forces each term to be equal to $n_{\binom{r-1}{2}+2}$. In particular, $I \cap B_u \cap A^1_{1v} = I \cap B_u \cap B_v$ has cardinality $n_{\binom{r-1}{2}+2}$ as claimed. ■

The conclusion is that the sets B_i, $i = r, r+1, \ldots, m$, form an $n_{\binom{r-1}{2}+1}$-uniform set system. Any two of these sets intersect in $n_{\binom{r-1}{2}+2}$ elements. The (uniform) Fisher inequality (see for example in [1]) implies that there are no more sets than the cardinality of the ground set. In our case this says

$$m - r + 1 \leq |I| \leq n_{\binom{r-1}{2}}.$$

This inequality shows that

$$m \leq n_{\binom{r-1}{2}} + r - 1 = r^{\binom{r+1}{2}-\binom{r-1}{2}} + r - 1 = r^{2\cdot r-1} + r - 1$$

and the proof of the theorem is finished. ■

References

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