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On the rotation distance of graphs

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Abstract

Let (x, y) be an edge of a graph G. Then the rotation of (x, y) about x is the operation of removing (x, y) from G and inserting (x, y') as an edge, where y' is a vertex of G. The rotation distance between graphs G and H is the minimum number of rotations necessary to transform G into H. Lower and upper bounds are given on the rotation distance of two graphs in terms of their greatest common subgraphs and their partial rotation link of largest cardinality. We also propose some extremal problems for the rotation distance of trees.

1. Introduction

In [1, 4] operations were introduced for measuring the distance between graphs of the same order and size. Here we investigate some questions confined to rotation distances. We continue the research initiated in [2] and propose extremal problems on tree distance graphs.

Let G be a simple undirected graph (no multiple edges and loops) and suppose that $(x, y)\in E(G)$ and $(x, y')\notin E(G)$. Then the *rotation* of (x, y) about x is the operation of removing (x, y) from G and inserting (x, y') as an edge.

If $(x, y)=e$ and $(x, y')=f$, then we denote the rotation above by $(y\,.\,x\,.\,y')$ or equivalently by $(e\,.\,x\,.f)$, and the graph obtained by $G(y\,.\,x\,.\,y')$ or $G(e\,.\,x\,.f)$. Formally,

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if $G'=G(e.x.f)$ then $V(G')=V(G)$ and $E(G')=E(G)-e+f$. We say that H can be obtained from G by rotation or G can be rotated into H, if there exists a rotation $r=(e.x.f)$ of G such that $G(r)\cong H$, where $\cong$ denotes isomorphism.

A simple graph of order n having m edges is called an (n, m)-graph. The *rotation graph* of (n, m)-graphs is defined with the set of all nonisomorphic (n, m)-graphs as the vertex set and (G, H) is an edge if and only if G can be rotated into H. The *rotation distance* $\varrho(G, H)$ between (n, m)-graphs G and H is defined to be the number of edges of a minimum length path in the rotation graph joining G to H, i.e., the minimum number of rotations necessary to transform G into H.

If the rotation distance of (n, m)-graphs G and H is d, then there is a sequence of (n, m)-graphs $G_0=G, G_1, \ldots, G_d=H$ and a sequence of rotations $r_i=(y_i.x_i.y_i')$, $i=1, \ldots, d$, satisfying $(x_i, y_i)\in E(G_{i-1})$, $(x_i, y_i')\notin E(G_{i-1})$, and $G_i\cong G_{i-1}(r_i)$, $1\leqslant i\leqslant d$. For the sake of simplicity one may assume that all G_i have the same set V vertices, and $G_i=G_{i-1}(r_i)$, $1\leqslant i\leqslant d$.

In Section 2 we give lower and upper bounds on the rotation distance between graphs. We show that if G and H are (n, m)-graphs then $\varrho(G, H)\geqslant m-t_{\max}(G, H)$, where $t_{\max}(G, H)$ is the maximum number of edges of a subgraph contained in both G and H (Proposition 2.1). Note that $t_{\max}(G, H)$ is the size of the *greatest common subgraph* of G and H investigated in [3, 4]. Examples which are 2-regular graphs show that this lower bound is sharp (Theorem 2.3).

In [4] it is proved that $\varrho(G, H)\leqslant 2(m-t_{\max}(G, H))$. We improve on this bound by introducing the notion of rotation links (Proposition 2.4). Sharp upper bounds are derived on the rotation distances of some special classes of graphs (Propositions 2.10–2.12).

In [2] the problem of characterizing *distance graphs* (i.e., induced subgraphs of rotation graphs) is investigated. A large family of distance graphs is presented there, and the question of whether every graph is a distance graph is proposed. We show that complete biparite graphs $K_{3,3}$ and $K_{2,p}$, $p\geqslant 1$, are distance graphs (Propositions 3.1 and 3.2).

A rotation of a tree that does not disconnect the tree is called a *tree rotation*. As far as we know, the notion of tree rotations appears first in [5] as a tool in enumerating labeled trees. The *tree distance* of trees G and H is defined to be the minimum number of tree rotations necessary to transform G into H and is denoted by $\tau(G, H)$. The *tree rotation graph* is defined to be the graph with the set of all nonisomorphic trees of order n as the vertex set and (G, H) an edge if and only if G can be rotated into H. Clearly, $\tau(G, H)$ is the distance between G and H in the tree rotation graph. We will see that the distance between trees in the rotation graph and that in the tree rotation graph may differ (Proposition 3.3). Then we show that $\tau(G, H)\leqslant 2\varrho(G, H)$ for every tree G and H of the same size (Theorem 3.4).

In Section 4 some properties of the tree rotation graph are investigated. In particular, we consider extremal problems related to finding certain large subgraphs in the tree rotation graph. We show that the maximum degree in the tree rotation graph is between $n(n-3)$ and $37n^2/48-O(n\log n)$ (Propositions 4.1 and 4.2). We prove that the size p of the maximum induced star satisfies $2n - o(n) < p < 2n - 2$ (Proposition 4.3). It is also shown that the diameter is $n-3$ and the radius is $n - o(n)$ (Propositions 4.7, 4.9 and 4.10).

2. General bounds

We establish lower and upper bounds on the rotation distance between two (n, m)-graphs.

Proposition 2.1. *If G and H are (n, m)-graphs, then*

$$\varrho(G, H) \geqslant m - t_{\max}(G, H).$$

Proof. Let $(G_0, r_1, G_1, r_2, \ldots, r_d, G_d)$ be a minimum path from G to H in the rotation graph, with $G_0 = G$, $G_d = H$ and $d = \varrho(G, H)$. For the sake of simplicity, assume that all graphs G_i have a common vertex set V, and $r_i = (e_i . x_i . f_i)$ is the rotation of G_{i-1} such that $G_i = G_{i-1}(r_i)$, $1 \leqslant i \leqslant d$.

Then the number d of rotations necessary to transform G_0 into G_d is at least as large as the number of edges of $G_0 - G_d$:

$$\varrho(G, H) \geqslant |E(G_0) - E(G_d)| = |E(G_0)| - |E(G_0 \cap G_d)| \geqslant m - t_{\max}(G, H). \qquad \square$$

A *rotation link* between (n, m)-graphs G and H with the same set of vertices is defined to be a bijection $\alpha : E(G) \Rightarrow E(H)$ such that $e \cap \alpha(e) \neq \emptyset$ for every $e \in E(G)$. A *partial rotation link* of cardinality k between G and H is the rotation link between two subgraphs $G' \subset G$ and $H' \subset H$ both having k edges.

From the proof of Proposition 2.1 we obtain the following immediate corollary.

Proposition 2.2. *Let G and H be (n, m)-graphs. Then*

$$\varrho(G, H) = m - t_{\max}(G, H)$$

if and only if there are graphs G' and H' with the same vertex set such that $G' \cong G$ and $H' \cong H$ and satisfying the following:

(a) $|E(G' \cap H')| = t_{\max}(G, H)$;

(b) *there exists a rotation link between $G' - H'$ and $H' - G'$.*

The result below gives an example where Proposition 2.2 yields the rotation distance.

Theorem 2.3. *If G and H are 2-regular graphs of order n, then*

$$\varrho(G, H) = n - t_{\max}(G, H).$$

Proof. Assume that G and H have vertex set V and $t_{\max}(G, H)=|E(G\cap H)|$. We will find a graph $H'\cong H$ such that $G'=G$ and H' satisfy (a) and (b) in Proposition 2.2.

Give a cyclic orientation to G, i.e., such that its cycles become directed cycles. We will define a cyclic orientation for H *compatible* with G such that each $e\in E(G\cap H)$ has the same sense in both G and H.

Let C be a cycle component of H and $D=G\cap C$. Thus $V(D)=V(G)\cap V(C)$ and $E(D)=E(G)\cap E(C)$. Note that D is either a cycle or the disjoint union of paths. If $D=G\cap C$ is a cycle, then orient the edges of H in C according to the orientation in G. If D is the disjoint union of paths, then the components of D can be closed by additional edges between the endpoints to get a cycle C' in the following way. The path components of D oriented as in G are to be arranged in an arbitrary cyclic sequence. Then join each endvertex to the first vertex of the next path by a directed edge. Since $t_{\max}(G, H)=|E(G\cap H)|$, $|E(D)|=|E(G)\cap E(C')|$ follows in both cases.

Doing the same steps for every cycle component of H we obtain a pair of graphs $G'=G$ and $H'\cong H$ that satisfy (a), i.e.,

$$|E(G'\cap H')|=|E(G\cap H)|=t_{\max}(G, H),$$

and such that H' has a cyclic orientation compatible with G'.

By the compatibility of the orientations, each vertex x is the tail of just one arc of G' and just one arc of H'. Let $\alpha(e)\in E(H'-G')$ be the unique arc with the same tail as e for every $e\in E(G'-H')$. Clearly, $\alpha: E(G'-H')\Rightarrow E(H'-G')$ is a bijection; thus there is a rotation link between $G'-H'$ and $H'-G'$, and so (b) holds.

Since $n=m$, and $G'=G$ and H' satisfy (a) and (b), $\varrho(G, H)=n-t_{\max}(G, H)$ follows by Proposition 2.2. □

Observe that any edge of a graph can be rotated into any nonedge in at most two steps. Thus $\varrho(G, H)\leqslant 2m$ for all (n, m)-graphs G and H. Based on this observation $\varrho(G, H)\leqslant 2(m-t_{\max}(G, H))$ was proved in [4], and the sharpness of the bound was established. This bound can be refined as follows.

Proposition 2.4. *Let G and H be (n, m)-graphs with the same set of vertices. If $T\subseteq G\cap H$ has t edges and there exists a partial rotation link of cardinality k between $G-T$ and $H-T$, then $\varrho(G, H)\leqslant 2(m-t)-k$.*

Proof. Let α be a partial rotation link of cardinality k between $G-T$ and $H-T$. Assume that α is defined on $\{e_1, \ldots, e_k\}$ and let $x_i\in e_i\cap\alpha(e_i)$, $1\leqslant i\leqslant k$. Then at most k rotations $(e_i . x_i . \alpha(e_i))$ for $e_i\neq\alpha(e_i)$, $i=1, \ldots, k$, and at most 2 more rotations for each edge of $E(G-T)\setminus\{e_1, \ldots, e_k\}$ yields that $\varrho(G, H)\leqslant 2(m-t-k)+k=2(m-t)-k$. □

In order to obtain an upper bound on the rotation distance between two graphs using Proposition 2.4, the maximum cardinality of a partial rotation link between the graphs is needed.

For $X, Y\subseteq V(G)$, set $m_G(X, Y)=|\{(x, y)\in E(G): x\in X, y\in Y\}|$.

Theorem 2.5. *Let G and H be two graphs with vertex set V and with both graphs having m edges. Then the maximum cardinality of a partial rotation link between G and H is*

$$k=2m-\max_{A\subseteq V}\{m_G(A,A)+m_H(V\setminus A,V\setminus A)\}.$$

Proof. The proof is based on the 'defect form' of Hall's theorem (cf. [7, 8]). Let U be a bipartite graph with bipartition (X, Y). Then the maximum cardinality of a matching from X into Y equals

$$|X|-\max_{X'\subseteq X}\{|X'|-|\Gamma(X')|\},$$

where $\Gamma(X')\subseteq Y$ is the set of all neighbors of the vertices in X'.

Now we define a bipartite graph U as follows. Let $X=E(G)$, $Y=E(H)$ and (e,f) is an edge of U for $e\in X$ and $f\in Y$ if and only if $e\cap f\neq\emptyset$. Clearly, the maximum cardinality k of a partial rotation link between G and H is equal to the maximum cardinality of a matching from X into Y. By the theorem above, this is

$$k=|X|-\max_{X'\subseteq X}\{|X'|-|\Gamma(X')|\}=m-\max_{G'\subseteq G}\{|E(G')|-|Y'|\},$$

where Y' is the set of all edges $f\in E(H)$ such that $f\cap V(G')\neq\emptyset$. Let $A=V(G')$. Then since $|Y'|=m-m_H(V\setminus A, V\setminus A)$, the equality becomes

$$k=m-\max_{A\subseteq V}\{m_G(A,A)-(m-m_H(V\setminus A,V\setminus A))\}$$

$$=2m-\max_{A\subseteq V}\{m_G(A,A)+m_H(V\setminus A,V\setminus A)\}. \qquad \square$$

By Proposition 2.4 and Theorem 2.5, we have that

$$\varrho(G,H)\leqslant 2(m-t)-k\leqslant 2m-k\leqslant\max_{A\subseteq V}\{m_G(A,A)+m_H(V\setminus A,V\setminus A)\},$$

which results in the following upper bound.

Corollary 2.6. *Let G and H be (n, m)-graphs. Then*

$$\varrho(G,H)\leqslant\min_{G',H'}\max_{A\subseteq V}\{m_{G'}(A,A)+m_{H'}(V\setminus A,V\setminus A)\},$$

where the minimum is taken over all G', H' with vertex set V such that $G'\cong G$ and $H'\cong H$.

In connection with Theorem 2.5 one may pose the question as to which pairs of graphs have rotation links. In passing, we mention some results motivated by this question.

Theorem 2.7. *Let G and H be two graphs, each with m edges, and the same vertex set V. Then, there is a rotation link between G and H if and only if $m_G(A,A)+m_H(V\setminus A,V\setminus A)\leqslant m$ for all $A\subseteq V$.*

Proof. If k is the maximum cardinality of a partial rotation link between G and H, then $k \leqslant m$. By Theorem 2.5,

$$2m - \max_{A \subseteq V} \{m_G(A, A) + m_H(V \setminus A, V \setminus A)\} = k \leqslant m;$$

consequently,

$$\max_{A \subseteq V} \{m_G(A, A) + m_H(V \setminus A, V \setminus A)\} \geqslant m.$$

Furthermore, observe that $k = m$ if and only if the last inequality is an equality which holds if and only if $m_G(A, A) + m_H(V \setminus A, V \setminus A) \leqslant m$ for every $A \subseteq V$. □

Corollary 2.8. *There exists a rotation link between two arbitrary trees with the same vertex set.*

Proof. We verify the sufficient condition in Theorem 2.7. Assume that $m_G(A, A) \geqslant m_H(V \setminus A, V \setminus A)$. If $m_H(V \setminus A, V \setminus A) = 0$, then $m_G(A, A) \leqslant m_G(V, V) = n - 1$; thus $m_G(A, A) + m_H(V \setminus A, V \setminus A) \leqslant n - 1 = m$. If both terms are positive, then, since the subgraphs of a tree are forests (or trees), $m_G(A, A) + m_H(V \setminus A, V \setminus A) \leqslant |A| - 1 + |V \setminus A| - 1 = |V| - 2 = m - 1$. □

Corollary 2.9. *There exists a rotation link two arbitrary k-regular graphs with the same vertex set.*

Proof. The sufficient condition in Theorem 2.7 holds, since $m_G(A, A) + m_H(V \setminus A, V \setminus A) \leqslant |A| k/2 + |V \setminus A| k/2 = |V| k/2 = m$ for every $A \subseteq V$. □

Let G and H be graphs with m edges having the same vertex set V. A set of distinct vertices $x_1, \ldots, x_p \in V$ is called a *common partial representative* of cardinality p for G and H if there exist pairwise distinct edges $e_i \in E(G)$ and pairwise distinct edges $f_i \in E(H)$ such that $x_i \in e_i \cap f_i$, $1 \leqslant i \leqslant p$.

Note that a common partial representative for G and H defines a partial rotation link between G and H if we let $\alpha(e_i) = f_i$.

Proposition 2.10. *If G and H are trees of order n which have a common subtree of t edges, then $\varrho(G, H) \leqslant n - 1 - t$.*

Proof. Assume that G and H have a common vertex set V and $G \cap H$ contains a tree T with t edges. Choose an arbitrary root $x \in V(T)$ for G and H. Since the edge set of a rooted tree of order n is represented by the $n-1$ vertices different from the root, the edges of T are represented with the same subset $V(T) \setminus \{x\}$ both in G and H. Consequently, $V \setminus V(T)$ is a common partial representative for $G - T$ and $H - T$. By Proposition 2.4, with $m = n - 1$ and $k = |V \setminus V(T)| = n - 1 - t$, we obtain that $\varrho(G, H) \leqslant 2(m - t) - k = n - 1 - t$. □

Proposition 2.11. *If G and H are connected graphs with $n \geqslant 3$ vertices and m edges, then $\varrho(G, H) \leqslant 2m-n$.*

Proof. Assume that both G and H are trees. Then $m=n-1$ and since any two trees have a common subtree of at least two edges (i.e., $t \geqslant 2$), we obtain, by Proposition 2.10, that $\varrho(G, H) \leqslant n-3<2m-n$.

Now assume that G and H are not trees. Then $G'=G-f$ is still connected for some $f \in E(G)$. Consider a spanning tree of G' rooted at a vertex incident with f. The edges of the spanning tree and f are n distinct edges of G represented by the vertices. Do the same with H and choose a common vertex set V for G and H. Since V is a common partial representative for G and H, by applying Proposition 2.4 with $T=\emptyset$ and $k=|V|=n$, we obtain that $\varrho(G, H) \leqslant 2m-n$. □

Proposition 2.12. *If G and H are simple graphs with n vertices and $m \geqslant p(p-1)/2$ edges, $p \geqslant 3$, then $\varrho(G, H) \leqslant 2m-p$. Moreover, this bound is sharp.*

Proof. First we show that G has p edges which can be represented with p distinct vertices. This is true for $p=3$. Let $p \geqslant 4$ and assume that the claim holds for smaller values. If G has a vertex of degree p, then the endvertices represent these edges. Thus we may assume that there is a nonisolated vertex x with degree at most $p-1$. Since $G-x$ has at least $p(p-1)/2-(p-1)=(p-1)(p-2)/2$ edges, the claim follows by induction. The same holds for H, i.e., it has p edges represented by p distinct vertices. Choose a common vertex set V for G and H such that p vertices represent p edges from G and from H.

Thus we have obtained that G and H have a common partial representative of cardinality p. Now by Proposition 2.4, with $T=\emptyset$ and $k=p$, $\varrho(G, H) \leqslant 2m-p$. □

The sharpness of the bound is shown by the following example. Let $m=p(p-1)/2$, $G=mK_2$ and H be the union of a clique of order p and p^2-2p isolated vertices. Then G cannot be transformed into H in less than $p^2-2p=2m-p$ rotations.

3. Distance graphs, tree rotations

A graph is called a *distance graph* if it is an induced subgraph of some rotation graph. It is not known whether every graph is a distance graph. This question was asked in [2], where a large family of distance graphs is presented.

Let $K_{p,q}$ denote the complete bipartite graph with p and q vertices in the partition classes.

Proposition 3.1. *$K_{3,3}$ is a distance graph.*

Proof. We will use the triple (a, b, c) of positive integers a, b and c to denote the tree formed by identifying an endvertex from each of three pairwise edge disjoint paths

4	9	2
3	5	7
8	1	6

Fig. 1. Chinese magic square.

with a, b and c edges. Such trees will be called 3-*rails*. Let $\mathscr{T}(n)$ be the set of all nonisomorphic trees of order n. We make the following observations:

(1) $(a, b, c) \in \mathscr{T}(n)$ if and only if $a+b+c+1=n$;

(2) $(a, b, c) \in \mathscr{T}(n)$ can be rotated into $(i, j, k) \in \mathscr{T}(n)$ if and only if $|\{a, b, c\} \cap \{i, j, k\}|=1$.

Consider a 3×3 magic square, e.g. the first one that was ever published in one of the famous books of Chinese mathematics in 1100 B.C. (see Fig. 1). The three numbers in every row and column add up to 15. Thus, by (1), the rows and columns encode 3-rails of $\mathscr{T}(16)$. Furthermore, according to (2), these 3-rails induce a $K_{3,3}$ in the rotation graph of $\mathscr{T}(16)$.

Proposition 3.2. *$K_{p,2}$ is a distance graph for every $p \geqslant 1$.*

Proof. Let $V=\{0, 1, \ldots, 2p+5\}$. Define G_1 with edge set

$$\{(i, i+1)\colon 1 \leqslant i \leqslant 2p+4\} \cup \{(0, 2j-1)\colon 1 \leqslant j \leqslant p+1\}.$$

Then G_1 has $n=2p+6$ vertices and $m=3p+5$ edges.

Let $r_j=(2j-1\,.\,0\,.\,2j)$, $1 \leqslant j \leqslant p$, be the rotations of G_1 and $H_j=G_1(r_j)$ for $j=1, \ldots, p$. One can easily see that $\varrho(H_i, H_j)=2$ for every $1 \leqslant i < j \leqslant p$. Thus $\{G_1, H_1, \ldots, H_p\}$ induces a star in the rotation graph of (n, m)-graphs.

Let $H'_j=H_j(r)$, $1 \leqslant j \leqslant p$, where $r=(2p+3\,.\,2p+2\,.\,1)$. It is easy to verify that $H'_i \cong H'_j$ for every $1 \leqslant i, j \leqslant p$. Moreover, the rotation distance of the graphs $G_2=H'_1$ and G_1 is equal to two. Thus $\{H_1, \ldots, H_p\} \cup \{G_1, G_2\}$ induces a $K_{p,2}$ in the rotation graph of (n, m)-graphs. □

The example below shows that the rotation distance of trees may increase when rotations disconnecting the graph are not allowed. We recall that a rotation of a tree that does not disconnect the tree is called a *tree rotation*. The *tree distance* of trees G and H, $\tau(G, H)$, is defined to be the minimum number of tree rotations necessary to transform G into H.

Let G and H be the trees given in Fig. 2, where the label at each vertex denotes the number of pendant edges incident to this vertex.

Proposition 3.3. *$\varrho(G, H)=2$, $\tau(G, H)=3$ for G and H in Fig. 2.*

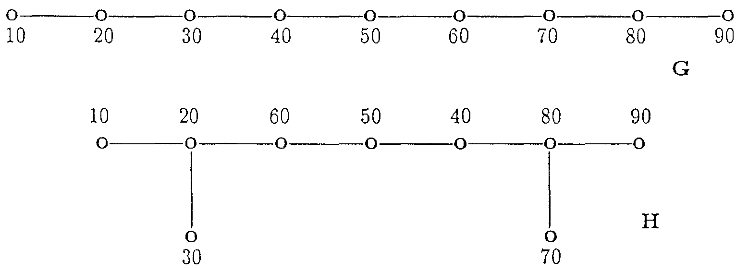

Fig. 2.

Proof. $\varrho(G, H) \geqslant 2$ is obvious, and the rotations $(30.40.80)$ and $(70.60.20)$ transform G into H. Thus $\varrho(G, H) = 2$ follows.

The tree rotations $(50.60.20)$, $(30.40.80)$ and $(70.60.50)$ transform G into H, therefore $\tau(G, H) \leqslant 3$. To show $\tau(G, H) > 2$, observe first that rotations of pendant edges need not be considered. Then $(30, 40), (60, 70) \in E(G)$ are to be replaced with $(20, 60), (40, 80) \in E(H)$. This is clearly impossible with two tree rotations. Thus $\tau(G, H) = 3$ as claimed. □

It is easy to see that for $d \geqslant 2$ allowing the intermediate rotations $r_1, \ldots, r_{d-1}$ to remove an edge (x, y) and insert a new edge (x, y') possibly parallel to an existing edge (x, y') (i.e., allowing $G_1, \ldots, G_{d-1}$ to be multigraphs), the distance of G and H does not decrease. As a consequence, edges can be rotated freely during intermediate steps. We refer to such rotations as *free rotations*. This technical advantage is used in the proof of the next result.

Theorem 3.4. $\tau(G, H) \leqslant 2\varrho(G, H)$ *for trees G and H of the same order.*

Proof. Let $d = \varrho(G, H)$, $S = (G_0, r_1, G_1, r_2, \ldots, r_d, G_d)$ be a minimum path from $G_0 = G$ to $G_d = H$, where all graphs have the same vertex set V and $r_i = (y_i . x_i . w_i)$ is a free rotation, $(x_i, y_i) \in E(G_{i-1})$, $G_i = G_{i-1}(r_i)$ for $1 \leqslant i \leqslant d$. Let $R = (r_1, \ldots, r_d)$ be the sequence of rotations.

Let k be the smallest integer, $1 \leqslant k \leqslant d$, such that G_j is a tree for every j, $k \leqslant j \leqslant d$. Then R has at most $k-1$ disconnecting rotations $(r_1, \ldots, r_{k-1})$. If $k = 1$, then each G_i, $1 \leqslant i \leqslant d$, is a tree, thus $\tau(G, H) = d$.

If $k > 1$, then $G_{k-1}(y_k . x_k . w_k) = G_k$ is a tree but G_{k-1} is not. Thus G_{k-1} has just two connected components. Denote their vertex sets by X and W, and let $x_k, y_k \in X$ and $w_k \in W$. Obviously, W induces a tree in G_{k-1} and the graph induced by X has just one cycle C containing the edge (x_k, y_k).

Now define m to be the smallest integer, $0 < m < k$, such that G_i contains C for every i, $m \leqslant i < k$. (One can say that when performing R to transform the tree G into the tree H, the 'very last cycle' C is 'closed' by r_m and it is 'broken' by r_k.) Observe that no rotation r_i, $m<i<k$, removes any edge of C, in particular, $(x_m, w_m)\in E(G_i)$ for every i, $m<i<k$.

Take the subsequence of R sending G_0 into G_{k-1}, then remove r_m and insert two rotations s_1, s_2 at the end:

$$R'=(r_1, \dots, r_{m-1}, r_{m+1}, \dots, r_{k-1}, s_1, s_2),$$

where $s_1=(y_m . x_m . w_k)$ and $s_2=(x_m . w_k . x_k)$. (If $m=k-1$ then $R'=(r_1, \dots, r_{k-2}, s_1, s_2)$.)

Since $x_k, x_m\in X$ and $w_k\in W$, the rotations s_1 and s_2 do not close C or any other cycle. Consequently, R' transforms G_0 into a tree G'_k. Then we can obtain the tree G_k from G'_k by two arbitrary tree rotations s_3 and s_4 sending (x_k, y_k) into (x_m, w_m). Notice that this is always possible, since x_m and w_m are in distinct components of $G'_k-(x_k, y_k)$.

Now replace the first k rotations of R with the sequence of $k+2$ rotations $(r_1, \dots, r_{m-1}, r_{m+1}, \dots, r_{k-1}, s_1, s_2, s_3, s_4)$, and let R'' denote the resulting sequence. Observe that the number of nontree rotations of R'' is at most $k-1$ (every rotation coming after s_1 is a tree rotation). By repeating this procedure to R'', and the sequences that result, at most $d/2$ times, we obtain a sequence of tree rotations with at most $4(d/2)=2d$ elements that transforms G into H. Thus $\tau(G, H)\leqslant 2\varrho(G, H)$. □

4. Extremal problems on the tree rotation graph

Let $\mathscr{T}(n)$ be the set of all nonisomorphic (unlabeled) trees of order n. Denote by $\mathscr{G}(n)$ the tree rotation graph of $\mathscr{T}(n)$ in which the vertex set is $\mathscr{T}(n)$ and two trees are joined by an edge if and only if they can be transformed into each other by a rotation. In this section we discuss certain extremal problems for $\mathscr{G}(n)$.

Let $T\in\mathscr{T}(n)$. The removal from T of an edge $e=(u, v)\in E(T)$ leaves two subtrees T' and T'' containing u and v, respectively. For every $e\in E(T)$ the number of tree rotations revolving this e is

$$|\{(u . v . x)\colon x\in V(T'-u)\}|+|\{(v . u . x)\colon x\in V(T''-v)\}|$$
$$=|V(T')-1|+|V(T'')-1|=n-2.$$

Thus the total number of tree rotations of T is at most $(n-1)(n-2)$.

This upper bound is slightly improved in the following proposition.

Proposition 4.1. *The maximum degree of $\mathscr{G}(n)$ is at most $n(n-3)$ for $n\geqslant 4$.*

Proof. The bound $(n-1)(n-2)$ can be decreased by exhibiting rotations which give no distinct neighbors of T in $\mathscr{G}(n)$. Let $T\in\mathscr{T}(n)$ and $x_1, x_2, \dots, x_p$ be a maximum path of T. If $p=3$, then T is a star which has degree one in $\mathscr{G}(n)$, so the claim is true.

Assume that $p\geqslant 4$. If one of x_2 and x_{p-1}, say x_2, is a vertex of degree two, then $T(x_2 . x_3 . x_1)\cong T$, i.e., the rotation $(x_2 . x_3 . x_1)$ gives no new neighbor of T. If x_{p-1} has degree at least three, then let (x_{p-1}, x_{p+1}) be an edge, clearly a pendant edge of T.

Then $T(x_{p-1}.x_{p+1}.y) \cong T(x_{p-1}.x_p.y)$ for every $y \in V(T) \setminus \{x_{p-1}, x_p, x_{p+1}\}$. Thus at least two rotations give the same neighbor of T in $\mathscr{G}(n)$.

In each case T has at most $(n-1)(n-2)-2=n(n-3)$ distinct neighbors. □

The next result shows that the maximum degree of $\mathscr{G}(n)$ is approximately n^2. The proof is omitted because of the technical difficulty of deciding whether two distinct rotations lead to isomorphic trees. This problem is related to reconstruction conjecture and the graph isomorphism problem (cf. [6, 9]).

Proposition 4.2. *Let $n=2^{k+1}+k$ and define the tree $T=T(k)$, $k \geqslant 1$, to be a path $(1, 2, \ldots, 2^{k+1})$ plus k pendant edges, one at each vertex 2^i, $i=1, \ldots, k$. Then T has $37n^2/48-\mathrm{O}(n \log n)$ neighbors in $\mathscr{G}(n)$.*

The upper bound for the maximum degree given in Proposition 4.1 yields that the order of maximum clique of $\mathscr{G}(n)$ is not greater than $n(n-3)+1$ for $n \geqslant 4$.

As far as the lower bound is concerned we only know of cliques of order $\mathrm{O}(n)$ in the tree rotation graph $\mathscr{G}(n)$. A clique of order $n-2$ can be obtained as follows. Let T be a path $(1, 2, \ldots, n-1)$ with one more pendant edge $(n-2, n)$. Then the rotations $(2.1.i)$ for $i=3, \ldots, n-1$ create $n-3$ distinct trees. It is easy to verify that each pair of these have rotation distance one.

Note that the size of the maximum clique of $\mathscr{G}(n)$ is probably not linear in n. In the proposition below, we show that the maximum size of an induced star of $\mathscr{G}(n)$ is $\mathrm{O}(n)$.

Propsoition 4.3. *Let p be the maximum size of an induced star of $\mathscr{G}(n)$. Then $2n-\mathrm{o}(n)<p \leqslant 2n-2$.*

Proof. The rotations of any (n, m)-graph involving a fixed vertex of a fixed edge clearly form a clique in the rotation graph. Therefore, an induced star in any rotation graph has size at most $2m$, so the upper bound follows.

Let $2^k+k \leqslant n \leqslant 2^{k+1}+k$ and let $T(k)$ be the caterpillar defined in Proposition 4.2. Let $T=T(k)$ if $n=2^{k+1}+k$, otherwise let T be the left subtree of $T(k)$ containing $n+1$ vertices minus the pendant edge at vertex 2^{k+1}. Let n' be the last vertex of the longest path of T starting at 1; n' is approximately $n-\log n$. Then $T(i+1.i.i+2)$ and $T(i.i+1.i-1)$, for $i=2, \ldots, n'-2$, are $2n-\mathrm{o}(n)$ pairwise independent trees in $\mathscr{G}(n)$. Thus $p>2n-\mathrm{o}(n)$. □

A *pruning order* of a tree G of order n is an ordering $x_1, x_2, \ldots, x_n$ of $V(G)$ such that the set $\{x_i, x_{i+1}, \ldots, x_n\}$ induces a subtree of G in which x_i has degree one, $1 \leqslant i<n$. In that pruning order (x_i, x_j) is called a *forward edge* if and only if $i<j$.

Recall that the tree distance $\tau(G, H)$ of $G, H \in \mathscr{T}(n)$ is the distance between G and H in the tree rotation graph $\mathscr{G}(n)$. We will use the following result.

Proposition 4.4. *Let G and H be trees of order n with the same vertex set which have a common pruning order. If $|E(G \cap H)|=t$, then $\tau(G, H) \leqslant n-1-t$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the common pruning order of G and H, let e_i and f_i, $1 \leqslant i < n$, be the forward edges at x_i of G and H, respectively. Define $r_i = (e_i . x_i . f_i)$ for $e_i \neq f_i$, $1 \leqslant i < n$.

For every $i = 1, \ldots, n-1$, perform rotation r_i or do nothing depending upon whether r_i is defined or not. This sequence of $n-1-t$ rotations transforms G into H and clearly no rotation disconnects the tree. □

We formulate an application of Proposition 4.4 that becomes useful when bounding the distance between two trees. It is generalization of Proposition 2.10.

Corollary 4.5. *Let G and H be trees of order n. If G and H have a common subtree of t edges, then $\tau(G, H) \leqslant n-1-t$.*

Proof. Assume that $E(G) \cap E(H)$ contains the common subtree T of t edges. Let $x_{n-t}, \ldots, x_n$ be a pruning order of T. Then, one easily obtains a labeling $x_1, \ldots, x_{n-t-1}$ for the remaining vertices of G and H such that $x_1, \ldots, x_n$ becomes a common pruning order for G and H. Hence $\tau(G, H) \leqslant n-1-t$ follows from Proposition 4.4. □

Next we give a lower bound on the tree distance of two trees in terms of their degree sequences.

Proposition 4.6. *Let G and H be trees of order n with degree sequences $g_1 \geqslant g_2 \geqslant \cdots \geqslant g_n$ and $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n$, respectively. Then*

$$\tau(G, H) \geqslant \frac{1}{2} \sum_{i=1}^{n} |g_i - h_i| \geqslant \max_{1 \leqslant i \leqslant n} \{|g_i - h_i|\}.$$

Proof. Observe that a tree rotation decreases by 1 and then increases by 1 the degree of two distinct vertices. The other degrees remain unchanged. Therefore,

$$\tau(G, H) \geqslant \frac{1}{2} \min_{\pi} \sum_{i=1}^{n} |g_i - h_{\pi(i)}|,$$

where the minimum is taken over all permutations π of $\{1, \ldots, n\}$. We show that the above minimum can be obtained for the identity permutation. Let π be an optimal permutation such that $i = \pi(i)$ for every $1 \leqslant i < p \leqslant n$, and $p \neq \pi(p)$. Let $p = \pi(q)$ for some $p < q \leqslant n$.

Now define $\pi'(i) = \pi(i)$ for every i, $1 \leqslant i \leqslant n$, different from p and q and let $\pi'(p) = p$ and $\pi'(q) = \pi(p)$.

We claim that π' is still optimal. Indeed it is easy to check from $p < q$ and $p < \pi(p)$ that the inequality

$$\begin{aligned} |g_p - h_{\pi(p)}| + |g_q - h_{\pi(q)}| &= |g_p - h_{\pi(p)}| + |g_q - h_p| \geqslant |g_p - h_p| + |g_q - h_{\pi(p)}| \\ &= |g_p - h_{\pi'(p)}| + |g_q - h_{\pi'(q)}| \end{aligned}$$

holds and thus the claim follows.

Applying the same procedure for the remaining indices we obtain that the identity is an optimal permutation. The second inequality of the proposition is trivial. □

Note that Proposition 4.6 is valid for arbitrary graphs as well.

The *diameter* of a graph is the maximum length of the shortest path between any pair of its vertices. The diameter of the tree rotation graph is $\max_{G, H \in \mathcal{T}(n)} \tau(G, H)$.

Proposition 4.7. *The diameter of* $\mathcal{G}(n)$ *is* $n-3$ *for* $n \geqslant 3$.

Proof. Since any two trees have a common subtree of at least two edges, $\tau(G, H) \leqslant n-1-t \leqslant n-3$, by Corollary 4.5. Moreover, this bound is sharp, as the example of a star and a path show. □

For $1 \leqslant i \leqslant n$ let P_i be a path of i vertices rooted at an endpoint; then a *balanced caterpillar* $C(i)$ is defined as a rooted caterpillar of order n obtained from P_i plus an appropriate number of pendant edges such that each vertex of P_i has degree $\lfloor (n-2+i)/i \rfloor$ or $\lceil (n-2+i)/i \rceil$.

Proposition 4.8. *The length of a maximum induced path of* $\mathcal{G}(n)$ *is at least* $n \log n - \mathrm{O}(n)$.

Proof (*outline*). We show that $\tau(C(i+1), C(i)) \geqslant (n-2)/(i+1)-1$, for $1 \leqslant i < n$. Let g_j and h_j be the jth largest degree of $C(i+1)$ and $C(i)$, respectively. Since $g_{i+1} \geqslant \lfloor (n-2)/(i+1) \rfloor + 1$ and $h_{i+1} = 1$, by Proposition 4.6,

$$\tau(C(i+1), C(i)) \geqslant |g_{i+1} - h_{i+1}| \geqslant \lfloor (n-2)/(i+1) \rfloor \geqslant (n-2)/(i+1) - 1.$$

Observe that $C(i+1)$ and $C(i)$ are distinct trees for each $i = 1, \ldots, n-4$, since $(n-2)/(i+1)-1 > 0$.

Let $M(i)$ be the set of trees inducing a path of minimum length in $\mathcal{G}(n)$ between $C(i+1)$ and $C(i)$, $1 \leqslant i \leqslant n-4$. Now the tree distance of each tree of $M(i) \setminus \{C(i), C(i+1)\}$ from each tree of $M(j) \setminus \{C(j), C(j+1)\}$ is more than one for $1 \leqslant i < j \leqslant n-4$. Thus the union of $M(i)$ for $1 \leqslant i \leqslant n-4$ is an induced path of length at least

$$(n-2) \sum_{i=2}^{n-3} (1/i - 1) = n \log n - \mathrm{O}(n). \qquad \square$$

Now we give upper and lower bounds on the *radius*, defined as the minimum length of the longest induced path starting from any vertex of the graph. The radius of the tree rotation graph is $\min_{T \in \mathcal{T}(n)} \max_{G \in \mathcal{T}(n)} \tau(T, G)$. Propositions 4.9 and 4.10 will show that the radius of $\mathcal{G}(n)$ is $n - \mathrm{o}(n)$.

Proposition 4.9. *Let* $n = k^2$, $k \geqslant 2$, *and* $C(\sqrt{n})$ *be the balanced caterpillar of order* n *obtained from the path* $P_{\sqrt{n}}$ *with* $\sqrt{n}-1$ *pendant edges added at each vertex. Then* $\tau(T, C(\sqrt{n})) \leqslant n - \sqrt{n}$ *for every tree* T *of order* n.

Proof. Suppose first that T has a vertex of degree at least $\sqrt{n}-1$. Then $T\cap C(\sqrt{n})$ contains a common star of $\sqrt{n}-1$ edges, hence by Corollary 4.5, $\tau(T, C(\sqrt{n}))\leqslant n-1-(\sqrt{n}-1)=n-\sqrt{n}$. Assume now that each vertex of T has degree less than $\sqrt{n}-1$. For the sake of simplicity, we will label the vertices of T and $C(\sqrt{n})$ with integers from 1 to n.

A pruning order of $C(\sqrt{n})$ is obtained by the following labeling. Label the vertices of the $\sqrt{n}$-path of $C(\sqrt{n})$ with $i\sqrt{n}, i=1,\dots,\sqrt{n}$, and for fixed i label the endvertices adjacent to $i\sqrt{n}$ with $(i-1)\sqrt{n}+j$, $j=1,\dots,\sqrt{n}-1$. We claim that T has a pruning order $1,\dots,n$ such that (p_i, q_i) is an edge for every $p_i=(i-1)\sqrt{n}+1$ and $q_i=i\sqrt{n}$, $i=1,\dots,\sqrt{n}$.

The labeling of T goes in $\sqrt{n}$ stages. In the ith stage, $1\leqslant i\leqslant\sqrt{n}$, label an endvertex with $p_i=(i-1)\sqrt{n}+1$ and label its neighbor with $q_i=i\sqrt{n}$. Now we prune vertex p_i, then $\sqrt{n}-2$ more endvertices first taking all neighbors of q_i, and we label them from the labels $(i-1)\sqrt{n}+j, j=2,\dots,\sqrt{n}-3$. Observe that q_i becomes an endvertex during this stage since the maximum degree of T is less than $\sqrt{n}-1$. Finally, we prune q_i. This pruning order of T has the required property.

We have shown that $1,\dots,n$ is a common pruning order of T and $C(\sqrt{n})$ with $\sqrt{n}$ common edges $(p_i, q_i)\in E(T\cap C(\sqrt{n}))$, $i=1,\dots,\sqrt{n}$. Then, by Corollary 4.5, $\tau(T, C(\sqrt{n}))\leqslant n-1-\sqrt{n}<n-\sqrt{n}$ follows. □

Proposition 4.10. *The radius of* $\mathcal{G}(n)$ *is at least* $(1-\varepsilon)n$ *for any* $\varepsilon>0$ *and sufficiently large* n.

Proof (*outline*). Assume T is an arbitrary tree with degree sequence $d_1\geqslant d_2\geqslant\cdots d_n$. Select $1>\alpha_1>\alpha_2>\cdots>\alpha_m>0$ such that $(m-1)\varepsilon\geqslant 4$. Let $T_i=C(n^{1-\alpha_{i+1}})$ be the balanced caterpillar defined above. (In this outline n powers are treated as integers.)

We claim that for some i, $\tau(T, T_i)\geqslant(1-\varepsilon)n$. Using Proposition 4.6, it is enough to show that for some i

$$\sum_{j\in A_i}(n^{\alpha_{i+1}}-d_j)\geqslant(1-\varepsilon)n, \tag{1}$$

where A_i, $1\leqslant i\leqslant m-1$, denotes the set of all indices $p\in\{1,\dots,n\}$ for which $d_p\leqslant n^{\alpha_{i+1}}$, and

$$n^{1-\alpha_i}=\frac{n}{n^{\alpha_i}}<p\leqslant\frac{n}{n^{\alpha_{i+1}}}=n^{1-\alpha_{i+1}}.$$

Assuming that (1) is false, for $i=1,\dots,m$ we obtain

$$\sum_{j\in A_i}d_j>\sum_{j\in A_i}n^{\alpha_{i+1}}-(1-\varepsilon)n.$$

By definition of A_i, $d_j>n^{\alpha_{i+1}}$ for $j\in B_i\setminus A_i$, where $B_i=\{n^{1-\alpha_i}+1,\dots,n^{1-\alpha_{i+1}}\}$.

Therefore,

$$\sum_{j\in B_i} d_j > \sum_{j\in B_i} -(1-\varepsilon)n.$$

Because $|B_i| = n^{1-\alpha_{i+1}} - n^{1-\alpha_i}$, we get

$$\sum_{j\in B_i} d_j > n - n^{1-\alpha_i+\alpha_{i+1}} - n + \varepsilon n = \varepsilon n - n^{1-\alpha_i+\alpha_{i+1}} \geqslant \varepsilon n/2 \tag{2}$$

if n is sufficiently large.

Adding (2) for $i = 1, \ldots, m-1$, the left-hand side is smaller than $2n$ but it is at least $(m-1)\varepsilon n/2$. Thus $4 > (m-1)\varepsilon$, contradicting the choice of m.

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