Rainbow Coloring the Cube

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ABSTRACT

We prove that for \( d \geq 4 \), \( d \neq 5 \), the edges of the \( d \)-dimensional cube can be colored by \( d \) colors so that all quadrangles have four distinct colors. © 1993 John Wiley & Sons, Inc.

At the recent 23rd Southeastern Conference on Graph Theory, Combinatorics, and Computing, Puhua Guan asked the following question: Is it possible to color the edges of the \( d \)-dimensional cube \( Q_d \) with \( d \) colors so that all quadrangles of \( Q_d \) are colored with four distinct colors? This makes sense only for \( d \geq 4 \) and Guan mentioned that he has constructed such a coloring for \( d = 4 \). In this article we give an affirmative answer to this question, except for \( d = 5 \), where the required coloring does not exist.

We call an edge-coloring of a graph \( G \) a rainbow coloring if the edges of every quadrangle (\( C_4 \) in what follows) of \( G \) are colored with distinct colors. Let \( rb(G) \) denote the minimum number of colors in a rainbow coloring of \( G \). Notice that \( rb(G) = 1 \) if \( G \) has no quadrangles, otherwise \( rb(G) \geq 4 \).

Rainbow colorings seem particularly interesting for graphs having the following property: any two incident edges are in a quadrangle of the graph.

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In this case, rainbow colorings are automatically proper edge colorings in the usual sense, i.e., each color class is the union of disjoint edges. Since this property is preserved under taking Cartesian products of graphs, it seems natural to study $rb(G \times H)$ in general. Although we focus our attention on $Q_d$, some lemmas are used that point to the more general setting.

Rainbow colorings are also related to total colorings. A coloring of edges and vertices (elements) of a graph is total if both edge and vertex colorings are proper and two elements of the same color are not incident. In what follows, $c(x)$ and $c(x, y)$ will be used to denote the color of a vertex $x$ and edge $xy$, respectively.

**Theorem 1.** If $d \geq 4, d \neq 5$ then $rb(Q_d) = d$.

**Corollary.** If $d \geq 3, d \neq 4$, there exists a total $(d + 1)$-coloring of $Q_d$, which is also a rainbow coloring.

**Proof.** Let $\chi$ be a rainbow $(d + 1)$-coloring of $Q_{d+1}$ from Theorem 1. Consider $Q_{d+1}$ as two disjoint copies of $Q_d$ with a factor between them. On one of these copies $\chi$ induces a rainbow $(d + 1)$-coloring and the colors of the factor edges give a proper vertex coloring on their end points in the copy of $Q_d$ in question. It is immediate that this coloring is total on $Q_d$.

Perhaps at this point it is useful to remark that it is easy to construct directly a total $(d + 1)$-coloring of $Q_d$ for $d \geq 3$ (without the additional rainbow property). This can be done by induction on $d$ as follows. To anchor the induction, take a total 4-coloring of $Q_3$ (see Figure 1). For the inductive step, take two disjoint copies of $Q_d$, say $A$ and $B$. Join the corresponding vertices of $A$ and $B$ by a factor $x_iy_i, i = 1, 2, \ldots, 2^d$.

Set $c(x_i, y_i) = d + 2$. Select any permutation $\Pi$ on the set $\{1, 2, \ldots, d + 1\}$ of colors that has no fixed point. Take a total $(d + 1)$-coloring on $A$ using colors $1, 2, \ldots, d + 1$ (induction) and permute colors on $B$ as follows:

$$c(y_i) = \Pi(c(x_i)), \quad c(y_i, y_j) = \Pi(c(x_i, x_j)).$$

This argument gives the following proposition.

**Proposition 1.** The total chromatic number of $Q_d$ is $d + 1$ for $d \geq 3$. 

![Figure 1](image-url)
The proof of Theorem 1 is based on three simple lemmas. Lemma 1 produces a rainbow coloring of the products of graphs from the rainbow coloring of its factors. It shows that \( rb(G_1 \times G_2) \leq rb(G_1) + rb(G_2) \) when \( G_1 \) and \( G_2 \) satisfy an additional condition. The other two lemmas are refinements of Lemma 1 for the special case when \( G_2 = Q_3 \) or \( G_2 = Q_2 \).

Before stating the main lemma, its key ingredient, the rainbow variant, is defined. Assume that \( G \) and \( H \) are isomorphic, \( H = f(G) \) under an isomorphism \( f \). Suppose also that both \( G \) and \( H \) have rainbow colorings. Then \( H \) and \( G \) are called rainbow variants (or simply variants) if

\[
    c(x, y) \neq c(f(x), f(y)) \quad \text{for all } xy \in E(G).
\]

A simple example of a rainbow variant is defined by a color-shift: if \( G \) is rainbow colored with colors 1, 2, \ldots, \( k \) then an \( i \)-shift is the variant \( H = f(G) \) equipped with the rainbow coloring \( c(f(x), f(y)) = c(x, y) + i \mod k \). A more complex example is shown in Figure 1, where the two variants are obtained by the automorphism of the cube that exchanges opposite corners. Unlike the shift, this variant does not preserve color classes under the isomorphism of the variants.

**Lemma 1.** Assume that \( G_1 \) has a rainbow \( p_1 \)-coloring, and \( G_2 \) has a rainbow \( p_2 \)-coloring. Furthermore, assume \( G_1 \) has a proper vertex \( q_1 \)-coloring, and \( G_2 \) has a proper vertex \( q_2 \)-coloring satisfying the "cross inequalities"

\[
    q_1 \leq p_2 \quad \text{and} \quad q_2 \leq p_1.
\]

Then \( rb(G_1 \times G_2) \leq p_1 + p_2 \).

**Proof.** Let \( S_1 \) and \( S_2 \) be disjoint color sets, \( |S_i| = p_i \) and fix a rainbow \( p_i \)-coloring on \( G_i \) with color set \( S_i \) (\( i = 1, 2 \)). Also fix proper vertex colorings of \( G_1 \) and \( G_2 \), using colors 1, 2, \ldots, \( q_1 \) and 1, 2, \ldots, \( q_2 \), respectively. Write \( G_1 \times G_2 \) as

\[
    \left( \bigcup_{x \in V(G_1)} G_2(x) \right) \bigcup \left( \bigcup_{y \in V(G_2)} G_1(y) \right)
\]

where \( G_2(x) \) is the copy of \( G_2 \) in \( G_1 \times G_2 \), which corresponds to \( x \); similarly, \( G_1(y) \) is the copy of \( G_1 \) in \( G_1 \times G_2 \), which corresponds to \( y \).

A rainbow coloring of \( G_1 \times G_2 \) is defined as follows. For each vertex \( x \in V(G_1) \), let \( G_2(x) \) be the \( c(x) \)-shift of \( G_2 \). Similarly, for each vertex \( y \in V(G_2) \), let \( G_1(y) \) be the \( c(y) \)-shift of \( G_1 \).

Observe that the shift-property and "cross-condition" (1) ensure that \( G_2(x) \) and \( G_2(x') \) are variants if \( c(x) \neq c(x') \). By the same reason, \( G_1(y) \) and \( G_2(y') \) are also variants if \( c(y) \neq c(y') \).
Since the vertex colorings are proper and \( S_1 \cap S_2 = \emptyset \), the coloring is a rainbow-coloring with \( p_1 + p_2 \) colors. \( \blacksquare \)

**Lemma 2.** If \( G_1 \) is bipartite with at least one edge then,

\[
rb(G_1 \times Q_3) = rb(G_1) + 3
\]

**Proof.** We proceed in the spirit of the proof of Lemma 1 with \( Q_3 \) replacing \( G_2 \). Define \( p_1 = rb(G_1) \), \( S_1 = \{5, 6, \ldots, p_1 + 4\} \), \( S_2 = \{1, 2, 3, 4\} \) and fix a rainbow \( p_1 \)-coloring and a proper vertex 2-coloring on \( G_1 \). Let \( A \) and \( B \) denote \( Q_3 \) with the edge and vertex colorings given in Figure 1.

For each vertex \( x \in V(G_1) \), let \( G_2(x) \) be either \( A \) or \( B \) depending on the color of \( x \). For each vertex \( y \in V(Q_3) \), let \( G_1(y) \) be the \( c(y) \)-shift of \( G_1 \). The coloring is a rainbow-coloring of \( G_1 \times Q_3 \) if \( p_1 \geq 4 \). However, \( p_1 + 4 \) colors are used instead of \( p_1 + 3 \).

To eliminate one color, say color 5, the special feature of the variants \( A \) and \( B \) is used that may be checked in Figure 1: the corresponding vertices of \( A \) and \( B \) miss the same color. This means that each edge of color 5 in \( G_1 \times Q_3 \) can be recolored to one of the colors in \( \{1, 2, 3, 4\} \) without violating the rainbow property. If \( p_1 \leq 3 \) then \( G_1 \) is has no \( C_4 \), implying \( p_1 = 1 \). In this case the same proof works if we start with the bipartite vertex coloring of \( Q_3 \). \( \blacksquare \)

**Corollary 2.** If \( G_1 \) is bipartite and has no \( C_4 \), then \( rb(G_1 \times Q_3) = 4 \).

**Proof.** Since \( rb(G_1) = 1 \), Lemma 2 can be applied. \( \blacksquare \)

The next lemma is similar to that of Lemma 2, but eliminates two colors (instead of just one) from a rainbow coloring of the product by \( rb(G_1) + 4 \) colors.

**Lemma 3.** Let \( G_1 \) be a graph with a rainbow coloring using \( p_1 \geq 2 \) colors and with its vertex set properly colored using four other colors, say 1, 2, 3, 4. Assume two of the \( p_1 \) colors, say 5 and 6, span a subgraph that consists of alternating cycles, each alternating cycle formed by passing (repeatedly) through the vertex classes 1, 2, 3, 4, 1 in this order. Then \( rb(G_1 \times Q_2) \leq rb(G_1) + 2 \).

**Proof.** The proof follows the same idea as the proof of Lemma 2. First, a rainbow coloring of \( G_1 \times Q_2 \) is given with \( p_1 + 4 \) colors as follows. Consider \( G_1 \) as defined in Lemma 2 with its rainbow \( p_1 \)-coloring, and assume that the color set \( S_1 \) in this rainbow coloring is \( S_1 = \{5, 6, \ldots, p_1 + 4\} \).

Let \( H_i \) (for \( i = 1, 2, 3, 4 \)) be the variants of \( Q_2 \) colored by the set of colors \( S_2 = \{1, 2, 3, 4\} \) as shown in Figure 2. A partial rainbow coloring
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The alternating cycle in colors 5 and 6 is also shown in Figure 2 in the case when $G_1 = Q_4$.

For each vertex $x \in V(G_1)$, let $Q_2(x) = G_2(x) = H_c(x)'$, where $c(x)$ is the color assigned to vertex $x$. For each vertex $y \in V(Q_2)$, let $G_1(y)$ be the $c(y)$-shift of $G$.

In a fashion similar to that in the proof of Lemma 2, we reassign each edge colored by 5 and 6 with an appropriate color selected from $\{1, 2, 3, 4\}$. For example, if an edge of $G_1(y)$ has color 5 and joins vertices labeled with color 1 in $H_2$ and $H_3$, then neither of these vertices is incident in their respective $H_i$ to color 4. This allows the edge with color 5 to be recolored with color 4. Continue in this fashion around the alternating cycles (colored with 5 and 6) in each $G_1(y)$, and reassign each edge a color from $\{1, 2, 3, 4\}$ so that the coloring remains rainbow. This eliminates colors 5 and 6 entirely and completes the proof.

An example when $G_1 = C_8$ demonstrates the procedure and is shown in Figure 3.

The proof of Theorem 1 now uses the three lemmas in the following way. Lemma 2 colors $Q_4$ (with $G_1 = Q_1$) and $Q_7$ (with $G_1 = Q_4$). Lemma 3
colors $Q_6$ (with $G_1 = Q_4$) and Lemma 2 colors $Q_9$ (with $G_1 = Q_6$). Then repeated applications of Lemma 1 colors all $Q_d$ for $d \geq 8$ and $d \neq 9$. We can show that the cube $Q_5$ has no rainbow coloring with five colors by a special case analysis. Since the argument is special and lengthy, it will not be given here.

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