Rainbow Coloring the Cube

R. J. Faudree
DEPARTMENT OF MATHEMATICAL SCIENCES
MEMPHIS STATE UNIVERSITY
MEMPHIS, TENNESSEE

A. Gyárfás
COMPUTER AND AUTOMATION INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY

L. Lesniak
DEPARTMENT OF MATH AND COMPUTER SCIENCE
DREW UNIVERSITY
MADISON, NEW JERSEY

R. H. Schelp
DEPARTMENT OF MATHEMATICAL SCIENCES
MEMPHIS STATE UNIVERSITY
MEMPHIS, TENNESSEE

ABSTRACT

We prove that for $d \geq 4$, $d \neq 5$, the edges of the d-dimensional cube can be colored by d colors so that all quadrangles have four distinct colors. © 1993 John Wiley & Sons, Inc.

At the recent 23rd Southeastern Conference on Graph Theory, Combinatorics, and Computing, Puhua Guan asked the following question: Is it possible to color the edges of the d-dimensional cube Q_d with d colors so that all quadrangles of Q_d are colored with four distinct colors? This makes sense only for $d \geq 4$ and Guan mentioned that he has constructed such a coloring for $d = 4$. In this article we give an affirmative answer to this question, except for $d = 5$, where the required coloring does not exist.

We call an edge-coloring of a graph G a *rainbow coloring* if the edges of every quadrangle (C_4 in what follows) of G are colored with distinct colors. Let $rb(G)$ denote the minimum number of colors in a rainbow coloring of G. Notice that $rb(G) = 1$ if G has no quadrangles, otherwise $rb(G) \geq 4$.

Rainbow colorings seem particularly interesting for graphs having the following property: any two incident edges are in a quadrangle of the graph.

Journal of Graph Theory, Vol. 17, No. 5, 607–612 (1993)
 CCC 0364-9024/93/050607-06

In this case, rainbow colorings are automatically proper edge colorings in the usual sense, i.e., each color class is the union of disjoint edges. Since this property is preserved under taking Cartesian products of graphs, it seems natural to study $rb(G \times H)$ in general. Although we focus our attention on Q_d, some lemmas are used that point to the more general setting.

Rainbow colorings are also related to *total colorings.* A coloring of edges and vertices (elements) of a graph is *total* if both edge and vertex colorings are proper and two elements of the same color are not incident. In what follows, $c(x)$ and $c(x, y)$ will be used to denote the color of a vertex x and edge xy, respectively.

Theorem 1. If $d \geq 4, d \neq 5$ then $rb(Q_d) = d$.

Corollary. If $d \geq 3, d \neq 4$, there exists a total $(d + 1)$-coloring of Q_d, which is also a rainbow coloring.

Proof. Let χ be a rainbow $(d + 1)$-coloring of Q_{d+1} from Theorem 1. Consider Q_{d+1} as two disjoint copies of Q_d with a factor between them. On one of these copies χ induces a rainbow $(d + 1)$-coloring and the colors of the factor edges give a proper vertex coloring on their end points in the copy of Q_d in question. It is immediate that this coloring is total on Q_d. ∎

Perhaps at this point it is useful to remark that it is easy to construct directly a total $(d + 1)$-coloring of Q_d for $d \geq 3$ (without the additional rainbow property). This can be done by induction on d as follows. To anchor the induction, take a total 4-coloring of Q_3 (see Figure 1). For the inductive step, take two disjoint copies of Q_d, say A and B. Join the corresponding vertices of A and B by a factor $x_i y_i$, $i = 1, 2, \ldots, 2^d$.

Set $c(x_i, y_i) = d + 2$. Select any permutation Π on the set $\{1, 2, \ldots, d + 1\}$ of colors that has no fixed point. Take a total $(d + 1)$-coloring on A using colors $1, 2, \ldots, d + 1$ (induction) and permute colors on B as follows:

$$c(y_i) = \Pi(c(x_i)), \qquad c(y_i, y_j) = \Pi(c(x_i, x_j)).$$

This argument gives the following proposition.

Proposition 1. The total chromatic number of Q_d is $d + 1$ for $d \geq 3$.

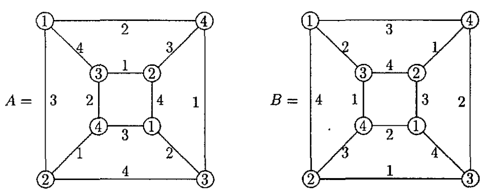

FIGURE 1

The proof of Theorem 1 is based on three simple lemmas. Lemma 1 produces a rainbow coloring of the products of graphs from the rainbow coloring of its factors. It shows that $rb(G_1 \times G_2) \leq rb(G_1) + rb(G_2)$ when G_1 and G_2 satisfy an additional condition. The other two lemmas are refinements of Lemma 1 for the special case when $G_2 = Q_3$ or $G_2 = Q_2$.

Before stating the main lemma, its key ingredient, the *rainbow variant,* is defined. Assume that G and H are isomorphic, $H = f(G)$ under an isomorphism f. Suppose also that both G and H have rainbow colorings. Then H and G are called *rainbow variants* (or simply variants) if

$$c(x,y) \neq c(f(x), f(y)) \quad \text{for all } xy \in E(G).$$

A simple example of a rainbow variant is defined by a *color-shift:* if G is rainbow colored with colors $1, 2, \ldots, k$ then an *i-shift* is the variant $H = f(G)$ equipped with the rainbow coloring $c(f(x), f(y)) = c(x,y) + i \pmod{k}$. A more complex example is shown in Figure 1, where the two variants are obtained by the automorphism of the cube that exchanges opposite corners. Unlike the shift, this variant does not preserve color classes under the isomorphism of the variants.

Lemma 1. Assume that G_1 has a rainbow p_1-coloring, and G_2 has a rainbow p_2-coloring. Furthermore, assume G_1 has a proper vertex q_1-coloring, and G_2 has a proper vertex q_2-coloring satisfying the "cross inequalities"

$$q_1 \leq p_2 \quad \text{and} \quad q_2 \leq p_1 . \tag{1}$$

Then $rb(G_1 \times G_2) \leq p_1 + p_2$.

Proof. Let S_1 and S_2 be disjoint color sets, $|S_i| = p_i$ and fix a rainbow p_i-coloring on G_i with color set S_i $(i = 1, 2)$. Also fix proper vertex colorings of G_1 and G_2, using colors $1, 2, \cdots, q_1$ and $1, 2, \cdots, q_2$, respectively. Write $G_1 \times G_2$ as

$$\left(\bigcup_{x \in V(G_1)} G_2(x) \right) \bigcup \left(\bigcup_{y \in V(G_2)} G_1(y) \right)$$

where $G_2(x)$ is the copy of G_2 in $G_1 \times G_2$, which corresponds to x; similarly, $G_1(y)$ is the copy of G_1 in $G_1 \times G_2$, which corresponds to y.

A rainbow coloring of $G_1 \times G_2$ is defined as follows. For each vertex $x \in V(G_1)$, let $G_2(x)$ be the $c(x)$-shift of G_2. Similarly, for each vertex $y \in V(G_2)$, let $G_1(y)$ be the $c(y)$-shift of G_1.

Observe that the shift-property and "cross-condition" (1) ensure that $G_2(x)$ and $G_2(x')$ are variants if $c(x) \neq c(x')$. By the same reason, $G_1(y)$ and $G_2(y')$ are also variants if $c(y) \neq c(y')$.

Since the vertex colorings are proper and $S_1 \cap S_2 = \phi$, the coloring is a rainbow-coloring with $p_1 + p_2$ colors. ∎

Lemma 2. If G_1 is bipartite with at least one edge then,

$$rb(G_1 \times Q_3) \leq rb(G_1) + 3$$

Proof. We proceed in the spirit of the proof of Lemma 1 with Q_3 replacing G_2. Define $p_1 = rb(G_1)$, $S_1 = \{5, 6, \ldots, p_1 + 4\}$, $S_2 = \{1, 2, 3, 4\}$ and fix a rainbow p_1-coloring and a proper vertex 2-coloring on G_1. Let A and B denote Q_3 with the edge and vertex colorings given in Figure 1.

For each vertex $x \in V(G_1)$, let $G_2(x)$ be either A or B depending on the color of x. For each vertex $y \in V(Q_3)$, let $G_1(y)$ be the $c(y)$-shift of G_1. The coloring is a rainbow-coloring of $G_1 \times Q_3$ if $p_1 \geq 4$. However, $p_1 + 4$ colors are used instead of $p_1 + 3$.

To eliminate one color, say color 5, the special feature of the variants A and B is used that may be checked in Figure 1: the corresponding vertices of A and B miss the same color. This means that each edge of color 5 in $G_1 \times Q_3$ can be recolored to one of the colors in $\{1, 2, 3, 4\}$ without violating the rainbow property. If $p_1 \leq 3$ then G_1 is has no C_4, implying $p_1 = 1$. In this case the same proof works if we start with the bipartite vertex coloring of Q_3. ∎

Corollary 2. If G_1 is bipartite and has no C_4, then $rb(G_1 \times Q_3) = 4$.

Proof. Since $rb(G_1) = 1$, Lemma 2 can be applied. ∎

The next lemma is similar to that of Lemma 2, but eliminates two colors (instead of just one) from a rainbow coloring of the product by $rb(G_1) + 4$ colors.

Lemma 3. Let G_1 be a graph with a rainbow coloring using $p_1 \geq 2$ colors and with its vertex set properly colored using four other colors, say $1, 2, 3, 4$. Assume two of the p_1 colors, say 5 and 6, span a subgraph that consists of alternating cycles, each alternating cycle formed by passing (repeatedly) through the vertex classes $1, 2, 3, 4, 1$ in this order. Then $rb(G_1 \times Q_2) \leq rb(G_1) + 2$.

Proof. The proof follows the same idea as the proof of Lemma 2. First, a rainbow coloring of $G_1 \times Q_2$ is given with $p_1 + 4$ colors as follows. Consider G_1 as defined in Lemma 2 with its rainbow p_1-coloring, and assume that the color set S_1 in this rainbow coloring is $S_1 = \{5, 6, \ldots, p_1 + 4\}$.

Let H_i (for $i = 1, 2, 3, 4$) be the variants of Q_2 colored by the set of colors $S_2 = \{1, 2, 3, 4\}$ as shown in Figure 2. A partial rainbow coloring

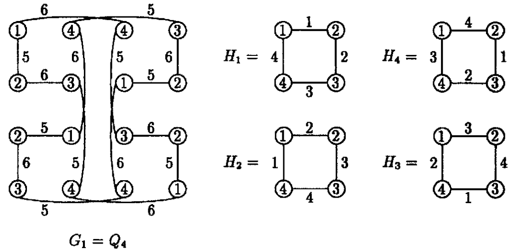

FIGURE 2

demonstrating the alternating cycle in colors 5 and 6 is also shown in Figure 2 in the case when $G_1 = Q_4$.

For each vertex $x \in V(G_1)$, let $Q_2(x) = G_2(x) = H_{c(x)}$, where $c(x)$ is the color assigned to vertex x. For each vertex $y \in V(Q_2)$, let $G_1(y)$ be the $c(y)$-shift of G.

In a fashion similar to that in the proof of Lemma 2, we reassign each edge colored by 5 and 6 with an appropriate color selected from $\{1, 2, 3, 4\}$. For example, if an edge of $G_1(y)$ has color 5 and joins vertices labeled with color 1 in H_2 and H_3, then neither of these vertices is incident in their respective H_i to color 4. This allows the edge with color 5 to be recolored with color 4. Continue in this fashion around the alternating cycles (colored with 5 and 6) in each $G_1(y)$, and reassign each edge a color from $\{1, 2, 3, 4\}$ so that the coloring remains rainbow. This eliminates colors 5 and 6 entirely and completes the proof.

An example when $G_1 = C_8$ demonstrates the procedure and is shown in Figure 3. ∎

The proof of Theorem 1 now uses the three lemmas in the following way. Lemma 2 colors Q_4 (with $G_1 = Q_1$) and Q_7 (with $G_1 = Q_4$). Lemma 3

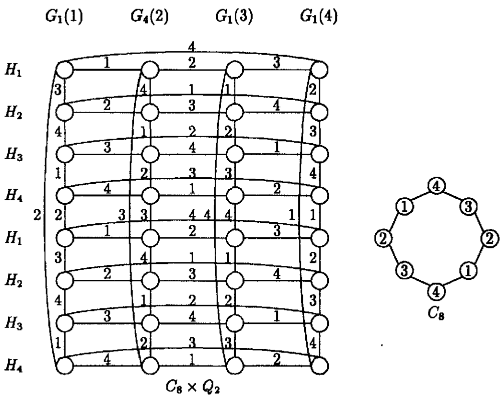

FIGURE 3

colors Q_6 (with $G_1 = Q_4$) and Lemma 2 colors Q_9 (with $G_1 = Q_6$). Then repeated applications of Lemma 1 colors all Q_d for $d \geq 8$ and $d \neq 9$. We can show that the cube Q_5 has no rainbow coloring with five colors by a special case analysis. Since the argument is special and lengthy, it will not be given here.

ACKNOWLEDGMENT

The research of RJF was supported by ONR Grant No. N00014-91-J-1085 and NSA Grant No. MDA 904-90-H-4034. AG is grateful for the hospitality of the Memphis State University during this research, which was supported by OTKA Grant of the Hungarian Academy of Sciences. The research of RHS was supported by NSA Grant No. MDA 904-89-H-2026. The research of LL was supported by ONR Grant No. N00014-93-1-0050.