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On-Line Graph Coloring and Finite Basis Problems

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Results and problems are announced concerning finite basis theorems in the theory of on-line colorings.

1. Introduction.

On-line graph coloring algorithms have been defined in [2] as algorithms whose input graph is presented vertex by vertex, at each step the current vertex is given with all adjacencies to previously given vertices. The algorithm must color irrevocably the current vertex and has to maintain a proper vertex coloring. The simplest and best understood example of an on-line coloring algorithm is the *First Fit* (or Greedy) algorithm, FF, which assigns the smallest proper color to the current vertex. The *chromatic number* of a graph G *with respect to an on-line coloring algorithm* A, $\chi_A(G)$, is the maximum number of colors used by A over all on-line presentations of G. Thus $\chi_A(G)$ measures the worst case behaviour of A on G. *The on-line chromatic number of* G, $\chi^*(G)$, is the minimum of $\chi_A(G)$ over all on-line algorithms A. Equivalently, the on-line chromatic number can be defined as the value of the following two person game. *Drawer* and *Painter* know graph G. Drawer presents the vertices of G one by one and Painter assigns a proper color to the current vertex. Drawer wants to force as many colors as possible, Painter wants to use as few colors as possible.

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Certain results in recursive combinatorics can be translated or closely related to on-line coloring results. For example, Kierstead and Trotter proved in [7] that the on-line chromatic number of any interval graph is at most $3\chi(G) - 2$ and for certain interval graphs this is best possible (χ stands for the usual chromatic number). Further examples and references are in [8] which is an excellent current survey paper with further references. We also recommend the whole volume [9] for on-line algorithms on several other problems.

The aim of this paper is to announce some results and problems of the authors related to possible finite basis theorems in on-line graph colorings.

We think our terminology is pretty standard and just a few special notations are needed before going further. The term *subgraph* is used to denote *induced subgraph.* Set

$$FF(k) = \{G : \chi_{FF}(G) \le k\}$$

and

$$OL(k) = \{G : \chi^*(G) \le k\}.$$

A graph G is *k-critical for on-line algorithm A* if $\chi_A(G) = k$ but $\chi_A(H) < k$ for all proper subgraphs H of G. A graph G is *on-line k-critical* if $\chi^*(G) = k$ but $\chi^*(H) < k$ for all proper subgraphs H of G. The complete graph on n vertices is denoted by K_n and the path on n vertices is denoted by P_n.

2. Finite basis theorems

An example of a finite basis result is the characterization of trees in $OL(k)$. This was done in [3] where the canonical tree T_k defined inductively as follows. Let T_1 be the one-vertex tree. The rooted tree T_k is defined by taking two disjoint copies of T_{k-1} and joining their roots with an edge e. The new root is an endpoint of e.

Theorem 1. ([3]) *A tree T is in $OL(k)$ if and only if T has no induced subtree isomorphic to T_{k+1}. Moreover, for any tree T, $T \in FF(k)$ if and only if $T \in OL(k)$.*

Theorem 1 shows that $FF(k)$ (or $OL(k)$) membership for trees depends on the presence of one subgraph. Similar result (for $FF(k)$) is true for the family of all graphs due to the following easy but useful result proved in [5].

Proposition 1. ([5]) *If G is k-critical for FF then G has at most 2^{k-1} vertices.*

Notice that equality is possible in Proposition 1. The tree T_k provides an example because of Theorem 1. Since $G \in FF(k)$ obviously if and only if G does not contain $(k+1)$-critical subgraphs for FF, Proposition 1 immediately gives a finite basis theorem for $FF(k)$.

Theorem 2. ([5]) *For fixed k, $FF(k)$ can be characterized by the exclusion of finitely many induced subgraphs (the $(k+1)$-critical graphs for FF).*

It is very easy to find explicitly the list of forbidden subgraphs in Theorem 2 for $k = 2$:

Proposition 2. *$G \in FF(2)$ if and only if G has no subgraphs isomorphic to K_3 or to P_4.*

Since both K_3 and P_4 are on-line 3-chromatic, we get that $FF(2) = OL(2)$ and this family is rather trivial, it contains the graphs whose components are complete bipartite graphs. For $k = 3$ it needs more work to determine the list of forbidden subgraphs. This was done in [5]:

Theorem 3. ([5]) *There are twenty two 4-critical graphs for FF. Sixteen of them are on-line 4-critical. The other six, G_i for $1 \le i \le 6$, are in $OL(3)$, these are shown on Figure 1.*

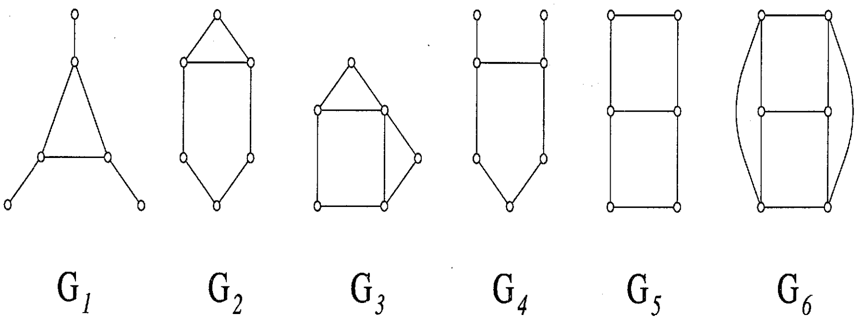

Figure 1. 4-critical graphs for FF which are on-line 3-chromatic.

Using Theorem 3, one can characterize the family $OL(3) - FF(3)$ in terms of forbidden subgraphs, by finding all on-line 4-critical graphs in $SUP(G_i)$ for $1 \le i \le 6$ where $SUP(H)$ denotes the family of graphs containing H as an induced subgraph. This program have been carried out in [4] and in [5] and summarized in what follows.

Let $4CR$ denote the family of on-line 4-critical graphs and let $4CR(i)$ stand for the subfamily $4CR \cap SUP(G_i)$ for $1 \le i \le 6$. It turns out that the simplest problem is to determine the family $4CR(1)$ since any one vertex extension of G_1 is in $4CR$ or contains a subgraph from $4CR$. As shown in [5], $4CR(1)$ has nine members, one of them is G_1 plus an isolated vertex. It is more difficult to find $4CR(i)$ for $2 \le i \le 4$. These families have been determined by a unified approach in [5], they together have finitely many (44, half of them disconnected) members. One reason why these families differ essentially from $4CR(1)$ is that while in $SUP(G_1)$ there is only one on-line 3-chromatic graph (G_1), this is not true for $SUP(G_i)$ for $2 \le i \le 6$. This difficulty culminates in case of $i = 5$. In fact, the whole paper [4] is devoted to describe the structure of a connected component containing G_5 in an on-line 3-chromatic graph. The results of that paper imply that $4CR(5)$ is also finite (with five connected and eight disconnected members). To state the most general finite basis theorem of [5], define a special family of graphs, $SPEC$, as graphs whose components are either G_6 or a graph G with $\chi_{FF}(G) = 3$ and both types of components are present.

Theorem 4. ([5]) *There are finitely many on-line 4-critical graphs which are not in $SPEC$.*

The results outlined above imply several other finite basis theorems. We select two of these.

Theorem 5. ([4]) *Bipartite on-line 3-chromatic graphs can be characterized by the exclusion of 12 forbidden induced subgraphs.*

It is worth noting that Theorem 5 can be paralleled by a similar result for triangle-free graphs with a longer list of forbidden subgraphs.

Theorem 6. ([5]) *Connected on-line 3-chromatic graphs can be characterized by the exclusion of 49 forbidden subgraphs.*

The finite basis theorems shown so far immediately imply that certain problems concerning on-line colorings are solvable in polynomial time. These are summarized in the next corollary.

Theorem 7. *The following problems can be solved in polynomial time:*

determine the on-line chromatic number of trees;

decide the First Fit k-colorability of graphs for fixed k;

decide the on-line 3*-colorability*
of bipartite graphs;
of triangle-free graphs;
of connected graphs;
and (as the most general case) of graphs not in $SPEC$.

For a long time the authors believed that a finite basis theorem exists for the class of all on-line 3-chromatic graphs. However, this turned out to be false, there are infinitely many on-line 4-critical graphs in $SPEC$. We define an infinite sequence of graphs S_k (for $k = 1, 2, \ldots$), such that $S_k \in SPEC$ for all k.

S_k will have $2k+1$ components, namely $A, B, D_1, C_2, D_2, C_3, D_3, \ldots, D_{k-1}, C_k, D'_k$ (see Figure 2).

Theorem 8. ([5]) *S_k is on-line* 4*-critical for all* k.

Since the graphs in Theorem 8 (and all other on-line 4-critical graphs in $SPEC$ known to the authors) are very special, we still believe that on-line 3-colorability can be checked in polynomial time for arbitrary graphs.

Conjecture 1. *Membership in* $OL(3)$ *can be checked in polynomial time for all graphs.*

Perhaps the complexity of on-line k-colorability changes at $k = 4$ like usual k-colorability does at $k = 3$.

Conjecture 2. *To check membership in* $OL(4)$ *is* NP*-complete.*

As Theorem 8 shows, there is no finite basis theorem for $OL(3)$. However, continuing the analogy with the off-line case, one can look for a finite basis result in the following sense. Although there are infinitely many off-line 3-critical graphs (the odd cycles), there are only finitely many for a fixed transversal number. Let $\tau(G)$, the *transversal number of* G, denote the minimum number of vertices in a graph G needed to meet each edge of G in at least one vertex. It is easy to observe that for each fixed k and t there are only finitely many off-line k-critical graphs with transversal number t (see in [1]). Probably this remains true for the on-line case but we can not prove it even for $k = 4$. Notice that the off-line result follows from the observation that an off-line k-critical graph has no equivalent vertices and

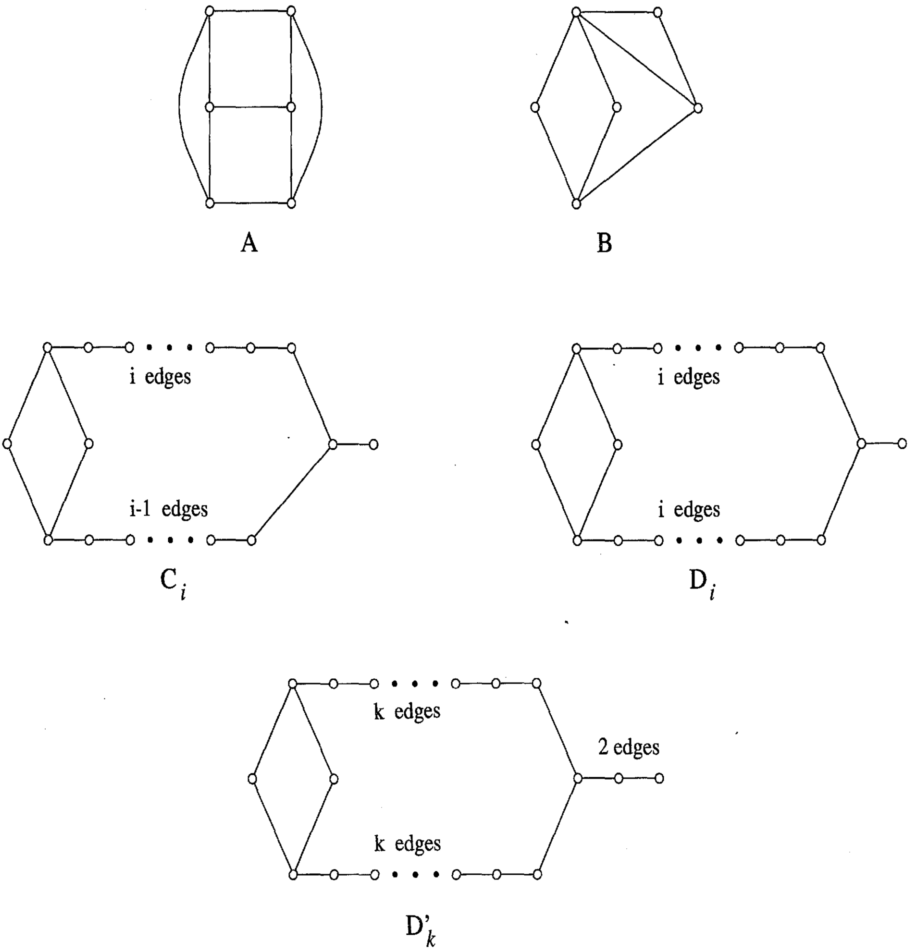

Figure 2. Component patterns for S_k.

isolated vertices. Neither statement remains true for the on-line case. A possibility to extend the off-line result is to prove the following conjecture (it also has independent interest).

Conjecture 3. *There exists a function $f(k)$ such that on-line k-critical graphs have no more than $f(k)$ equivalent vertices. In particular, on line k-critical graphs have no more than $f(k)$ isolated vertices.*

We finish by a problem which perhaps also related to finite basis problems. How many colors do we need to color on-line 3-chromatic graphs on-line? This seemingly senseless question refers to a subtle point in the definition of the on-line chromatic number of a family of graphs $\mathcal{F}$. In [3] the on-line chromatic number of $\mathcal{F}$, $\chi^*(\mathcal{F})$, have been defined as the value of the two person game of Drawer and Painter when both of them know $\mathcal{F}$ and Drawer presents on-line some member of $\mathcal{F}$ (but, of course, Painter does not know which one). In particular, when $\mathcal{F} = OL(3)$, the on-line chromatic number of $\mathcal{F}$ is the minimum number of colors needed to color an unknown on-line 3-chromatic graph on-line. It was shown in [3] that three colors are not enough even against two graphs (one is G_5 and the other is a $K_{1,3}$ with two subdivided edges). In [6] an easy argument is given to show that $\chi^*(OL(3))$ is bounded by 16, and using results of [4] and [5] we could determine it exactly.

Theorem 9. $\chi^*(OL(3)) = 4$.

Remark. At the time of submitting this paper we learned that K. Kolossa proved (private communication) that $\chi^*(OL(3)) \leq 5$ and she thinks her method may also give $\chi^*(OL(3)) = 4$.

We think that a similar result is true for $OL(k)$.

Conjecture 4. *There exists a function $f(k)$ such that $\chi^*(OL(k)) \leq f(k)$.*

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