Networks Communicating for Each Pairing of Terminals

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Let G be a multigraph of maximum degree Δ and with a set of t vertices of degree one, called terminals. We call G a (Δ, t)-*network* if for any pairing of its terminals there exist edge-disjoint paths in G between those pairs (t is even). The concept of (Δ, t)-networks is introduced to model the situation when switching processors having Δ ports are to be connected in such a way that simultaneous communication is possible for any pairing of the free ports. We establish some properties of (Δ, t)-*networks*. In particular, we investigate optimal (or near-optimal) networks and obtain lower and upper bounds on the function $n(\Delta, t)$, the minimum number of interior nodes a (Δ, t)-*network* can have.

1. INTRODUCTION

A processor is a node with Δ distinct I/O lines. Each of the Δ lines can be used to connect the processor to another processor or to a terminal. We wish to build a network from these processors so that it is possible to communicate simultaneously for any pairing of its terminal nodes. What is the minimum

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number of processors $n(\Delta, t)$ needed in such a network with t terminal nodes (t is an even positive integer)? Equivalently, for n processors, what is the maximum $t = t(\Delta, n)$ for which there exists such a network with t terminal nodes? The problem is to determine or estimate $n = n(\Delta, t)$ and to design optimal (or near-optimal) networks.

To describe the problem more precisely, we introduce the notion of a (Δ, t)-*network*. Let G be a multigraph with maximum degree Δ, with vertex set $V(G) = T(G) \cup I(G)$, where $|T(G)| = t$(t even) and the vertices of $T(G)$ are all of degree 1. Call G a (Δ, t)-network if for any pairing of the vertices of $T(G)$ there exist edge-disjoint paths in G between the paired vertices. To exclude trivial cases, it is always assumed that $\Delta \geq 3$, $t \geq 4$, and $t > \Delta$. If no confusion occurs, we use $T = T(G)$ and $I = I(G)$. T is referred to as the set of *terminal nodes* or *terminals* and I is called the set of *interior nodes* of the network.

We would like to point out that the notion of a (Δ, t)-network emerged from a practical problem which the first author met in the course of building a packet switching data network node from a given miniswitch having a limited number of ports. The miniswitch can establish virtual circuits between any pairs of its ports. The problem was how to connect a number of those switches together to have a larger capacity switch having more ports than the building element but keeping the basic feature of the miniswitch. The basic feature is that there is a given throughput in terms of packet per second for all of its ports and the total throughput is equal to the sum of the throughputs of all of its ports. Within these limitations, a packet entering the switch can leave at any other port. The ports are symmetric as far as the inbound and outbound packet traffic is concerned. By the heuristic approach, we found some practical solutions, for instance, from a miniswitch having 10 ports, we could put together seven pieces to form a switch having 24 ports. [The solution was the (10, 24)-network in which the inner vertices form a K_7 and four terminals are adjacent to all but one vertex of K_7.]

In a (Δ, t)-network, pairs of vertices of a graph are to be connected with edge-disjoint paths, thus the notion is clearly related to multicommodity flow problems. (A survey article on this is [11].) The concept is also related to linked and weakly linked (in our case $t/2$-linked) graphs (see [5–7]). These connections become clear by introducing the notion of a demand graph.

Given a (Δ, t)-network, let $C \subseteq I$ be the set of all vertices of the network adjacent to at least one terminal node. Assume that $C = \{c_1, \ldots, c_k\}$ and c_i is adjacent to $t_i (\geq 1)$ terminal nodes ($\Sigma_{i=1}^{k} t_i = t$). To a pairing of T, a *demand graph* D is associated, having vertices $d_1, \ldots, d_k$ and $d_i d_j$ being an edge of D with multiplicity m if there are m pairs of T between terminals adjacent to c_i and c_j. The demand graph may have (multiple) loops.

Clearly, the vertex d_i in D has degree t_i. Loosely speaking, it is straightforward that a network G with maximum degree Δ and with $t = \Sigma_{i=1}^{k} t_i$ terminals is a (Δ, t)-network if and only if *all* demand graphs with degree sequence $t_1, \ldots, t_k$ is realizable by edge-disjoint paths in $G_{[I]}$. For each demand graph, we have an instance of the edge-disjoint paths problem (see [4]). Figure 1 pictures an (8, 18)-network G built on the complete graph K_6, together with a demand graph.

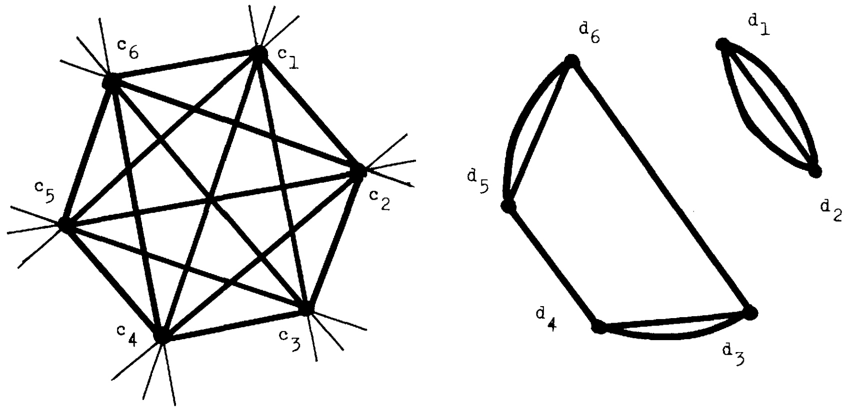

FIG. 1. An (8, 18)-network with a demand graph.

Define $n(\Delta, t)$ as the minimum number of interior nodes in a (Δ, t)-network. The (Δ, t)-networks attaining this minimum are called optimal networks. In fact, very few optimal networks are known, even for small values of Δ and t. Some of these networks are displayed in Figure 2. Perhaps, at first sight, it is not obvious that $(3, t)$-networks exist for arbitrary t, but it is easy to see that $n(3, t) \leq t^2$ (Proposition 1).

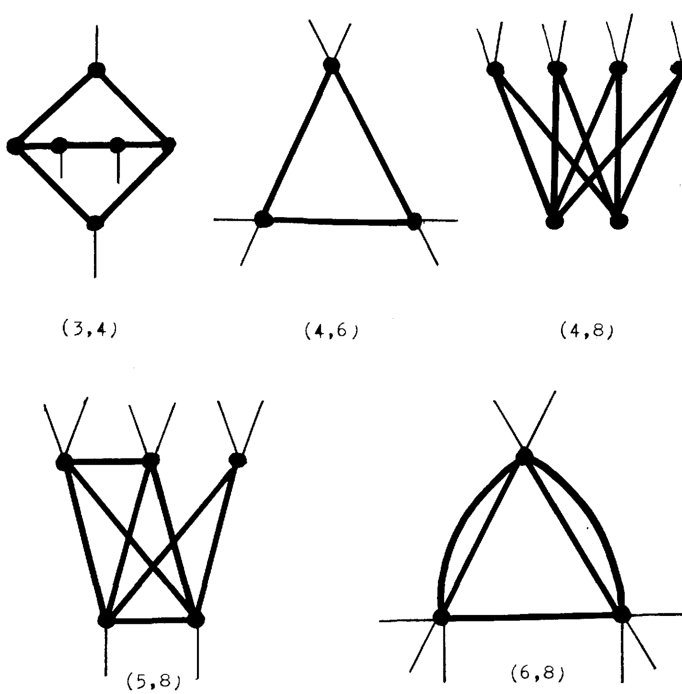

FIG. 2. Some optimal networks.

It will be shown that $n(\Delta, t) \geq (t/\Delta)\,[log_{\Delta-1}t]$ (Theorem 2), but we were not able to construct networks with so few vertices. Using k-dimensional grids and a simple substitution principle, it is possible to construct (Δ, t)-networks with $2t^{1+1/k}$ vertices for any positive integer k and for sufficiently large t (Section 5).

It seems interesting to investigate the network K_p^q where the set of interior nodes induces a clique K_p and there are q terminals attached to each node of K_p. We prove in Section 6 that the largest q such that K_p^q is a $(p + q - 1, pq)$-network satisfies $p/7.5 \leq q \leq p/2$ (Theorem 4 and Proposition 3), and our conjecture is that the upper bound is close to the truth (Conjecture 2).

The main problem left open is the order of magnitude of $n(3, t)$. The truth is probably close to the lower bound $\alpha t \log t$ (see Theorem 2). In fact, a good upper bound would follow from the following conjecture. Assume that $x_1, x_2, \ldots, x_{2^d}$ are the vertices of the d-dimensional cube and d is odd. Then, there exist edge disjoint paths from x_i to x_{i+1}, for $i = 1, 3, \ldots, 2^d - 1$. This is true for $i = 1, 3, \ldots, 2d - 1$ (i.e., for d pairs) as proved in [9]. As was reported by A. Frank recently, a very similar "path pairable" property was proposed for the symmetrically directed cubes.

From a practical point of view, it is useful to see how to make the routing algorithmic if the pairing of terminals is given. Routings on grid networks depend on good edge-coloring algorithms, but in the case of networks built on complete graphs, a greedy-type algorithm is used.

2. CUT BOUNDS

Assume that G is a (Δ, t)-network with $V(G) = I \cup T$, and I is partitioned into two sets A and B. Then, T is also partitioned into T_A and T_B according to the adjacencies of these vertices in I. Denote by $e(A, B)$ the set of all edges between A and B. It is immediate to check the

Cut Condition. For any (Δ, t)-network,

$$|e(A, B)| \geq \min(|T_A|, |T_B|).$$

However, the Cut Condition is not sufficient. Two simple examples are shown in Figure 3 (the first one is a fairly standard example). A more general example is the network built on the d-dimensional cube Q^d, for d even. To see this, consider the demand graph on the cube formed by the 2^{d-1} pairs of opposite corners. Since the shortest path between opposite corners has length d, the union of the edge-disjoint paths realizing the demand graph must cover all of the $d2^{d-1}$ cube edges exactly once. This is possible only for d odd, since for any vertex v just one path starts at v. However, Q^d satisfies the Cut Condition for every d, which follows by a result due to L. H. Harper, A. J. Bernstein, and S. Hart (see in [1]).

It follows easily from the Cut Condition that $n(\Delta, t) \geq 2t/\Delta$. More careful analysis of the Cut Condition gives the following improvement:

Theorem 1. $n(\Delta, t) \geq (5/2)(t/\Delta) - 1.$

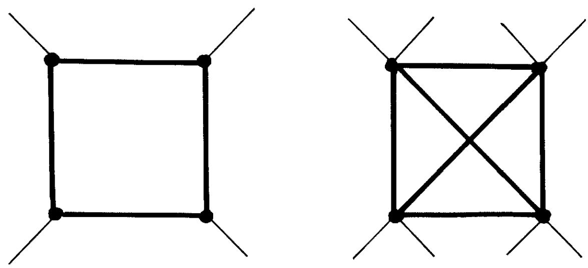

FIG. 3. Networks whose even multiples work.

Proof. It is known (see, e.g., in [8]) that in a graph of n vertices and m edges the average of an even cut (with sets of order $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$) is $m(\lceil n^2/4 \rceil)/\binom{n}{2}$. Thus, the graph has an even cut containing at most $(nm)/[2(n-1)]$ edges. Applying this observation for the graph $G_{[I]}$ induced by I in a (Δ, t)-network G, there exists an even cut (A, B) in $G_{[I]}$ such that $(nm)/[2(n-1)] \geq e(A, B)$. Let $t^* = \min(|T_A|, |T_B|)$. Since $m \leq (n\Delta - t)/2$, we obtain by the Cut Condition that

$$t^* \leq e(A, B) \leq \frac{nm}{2(n-1)} \leq \frac{n}{2(n-1)} \frac{n\Delta - t}{2}. \tag{1}$$

On the other hand, each vertex of I is adjacent to at most $\Delta/2$ vertices of T; therefore,

$$t^* + \left\lceil \frac{n}{2} \right\rceil \frac{\Delta}{2} \geq t. \tag{2}$$

The theorem follows by eliminating t^* from (1) and (2). ■

3. THE DISTANCE BOUND

Let G be a (Δ, t)-network, $x \in T$ and p be an integer $(p \geq 2)$. Obviously, there are at most $(\Delta - 1)^{p-1}$ vertices of T at a distance at most p from x. (The distance is understood to be in G.) Assuming that $t \geq 2(\Delta - 1)^{p-1}$, T contains at least $t/2$ terminals, each at distance $p + 1$ or more from x. Repeating this argument for all $x \in T$, one can see easily that there is a pairing of vertices of T such that the distance between each pair is at least $p + 1$ in G. The edge-disjoint paths for this pairing have at least $t(p + 1)/2$ edges. On the other hand, these paths use at most $|E(G)| \leq (|I|\Delta + t)/2$ edges, since each vertex of I has degree at most Δ in G. Thus, $t(p + 1) \leq 2|E| \leq |I|\Delta + t$, i.e., $|I| \geq tp/\Delta$. Now $t \geq 2(\Delta - 1)^{p-1}$ can be assured by setting $p = \lfloor \log_{\Delta-1} t/2 \rfloor + 1 \geq \lfloor \log_{\Delta-1} t \rfloor$, and we get

Theorem 2. $n(\Delta, t) \geq (t/\Delta) \lfloor \log_{\Delta-1} t \rfloor$.

Note that for small values of t ($t \le (\Delta - 1)^{2.5}$) the cut bound is better than that in Theorem 2.

4. PRINCIPLES OF BUILDING NETWORKS

4.1. Substitution

Assume that G is a (Δ, t)-network and $x \in I$ has degree $d_G(x) = p$. Then, one can substitute for x any (Δ', p)-network with $\Delta' < \Delta$ and the new network is still a (Δ, t)-network. Applying repeated substitutions for the vertices of a (Δ, t)-network G, one can define a (Δ', t)-network H for any $\Delta' < \Delta$ $(3 \le \Delta')$ such that $T(G) = T(H)$. If this principle is applied to a (Δ, t)-network G in which all vertices of $I(G)$ have degree Δ so that a (Δ', Δ)-network G' is substituted for each vertex of $I(G)$, then the result is a (Δ', t)-network H with $|I(H)| = |I(G)||I(G')|$.

4.2. Multiplication

A natural network operation on a (Δ, t)-network G is to define mG by replacing each edge in I by m parallel edges and replacing each terminal node x with m terminals attached to the same interior node as x. It seems plausible that mG is an $(m\Delta, mt)$-network—at least we do not know any counterexample.

Problem 1. If G is a (Δ, t)-network, is it true that mG is an $(m\Delta, mt)$-network?

The reverse implication is certainly false. Figure 3 shows two examples of a graph G for which mG is an $(m\Delta, mt)$-network iff m is even.

It is worth noting that Problem 1 has an affirmative answer when there are just two terminals at each interior node of G. Indeed, in this case, a demand graph D associated to a pairing of the terminals of mG is a $2m$-regular graph. Now by Petersen's theorem [10], the edge set of D has a partition into m sets defining 2-regular subgraphs $D_1, \ldots, D_m$ of D. Since G is a (Δ, t)-network, every D_i considered as a demand graph has a realization by edge-disjoint paths in a copy G_i of G, for $i = 1, \ldots, m$. Then, by identifying the corresponding vertices of $G_1, G_2, \ldots$ and G_m, we obtain a realization of D in mG.

4.3. Path Pairable Graphs

A graph of order $2n$ is called *path pairable* if for any ordering $v_1, \ldots, v_{2n}$ of its vertices there exist n edge disjoint paths from v_{2i-1} to v_{2i}, $i = 1, \ldots, n$. This notion is motivated by the obvious fact that a path pairable graph G of order $2n$ and maximum degree Δ defines a $(\Delta + 1, 2n)$-network by hanging a terminal node on each vertex of G. Path pairable graphs are studied in [2, 3]. (A nontrivial example of a path pairable graph is the Petersen-graph.) Path pairable graphs with a small maximum degree would provide a good upper bound on

$n(\Delta, t)$, the minimum number of interior nodes a (Δ, t)-network can have. A natural candidate is the d-cube Q^d for d odd (cf. Section 2).

Conjecture 1. The d-dimensional cube Q^d is path pairable for every odd d.

This is proved for $d = 3$ and supported by our result that for odd d the edge set of Q^d has a partition into paths of length d so that the endpoints of each path are opposite corners. For our purpose, it would be useful to answer the following weaker question:

Problem 2. Is the multicube cQ^d (each edge of Q^d is replaced with c parallel edges) path pairable for all d with some fixed integer c?

The significance of this problem (or, in fact, the existence of any path pairable graph with small degrees) is that it helps to construct nearly optimal networks. To illustrate this, assume that cQ^d is path pairable. Let G be the network with one terminal attached to each vertex of cQ^d [G has 2^d terminals and 2^d interior nodes, and it is $(cd + 1)$-regular]. Clearly, G is a $(cd + 1, 2^d)$-network. Substitute each interior node by an optimal $(3, cd + 1)$-network with $n(3, cd + 1)$ interior nodes. The resulting $(3, 2^d)$-network has $2^d n(3, cd + 1)$ interior nodes, which is certainly smaller than $2^d(cd + 1)^2$, using a simple observation (see Proposition 1). Thus, one would get $n(3, t) \leq \alpha t \log^2 t$, which is close to the lower bound $O(t \log t)$ in Theorem 2.

4.4. Asymptotic Structure

The cut condition shows that in a (Δ, t)-network each interior node is adjacent to at most $\lfloor \Delta/2 \rfloor$ terminal nodes. This implies that in a $(3, t)$-network the t terminals are adjacent to t distinct interior nodes. We could not decide which (if any) of the following two possibilities describe the asymptotic terminal connections of optimal (Δ, t)-networks: If Δ is fixed and t is large, then for an optimal (Δ, t)-network G

(a) the t terminal nodes of G are adjacent to t distinct interior nodes of G;
(b) the t terminals of G are partitioned into groups of $\lfloor \Delta/2 \rfloor$ nodes and each group is attached to a common interior node of G.

5. GRID NETWORKS

In this section, we prove that $n(3, t) \leq 2t^{1+\varepsilon}$, for any $\varepsilon > 0$ and for sufficiently large t. Clearly, the same bound is valid for any $\Delta \geq 3$. The idea is to construct an (m, m^k)-network H with $c_k m^k$ vertices and then use the substitution principle by replacing each vertex of H by a $(3, m)$-network with "few" vertices, e.g., with m^2 vertices shown by the next proposition (cf. Sections 4.1 and 4.3).

Proposition 1. $n(3, m) \leq m^2$.

Proof. Consider m vertical lines starting at terminal nodes $1, 2, \ldots, m$ and join line i and j with a horizontal segment so that each of these $\binom{m}{2}$ segments are at different heights. Define a network N with the $m^2 - m$ distinct endpoints of the segments as interior nodes and with the set of all vertical and horizontal segments between nodes as edges. It is easy to see that N is a $(3, m)$-network. ■

Let $B_k(m)$ denote the set of points of the k-dimensional grid of size m, i.e.,

$$B_k(m) = \{(x_1, x_2, \ldots, x_k)\text{: every } x_i \text{ is an integer, } 1 \leq x_i \leq m\}.$$

A *line* in $B_k(m)$ is a set of m points given by fixing $k - 1$ of the coordinates. Our purpose is to create a bipartite graph $G_k(m)$ by adding an independent set of $c_k m^{k-1}$ vertices to $B_k(m)$, each vertex adjacent to m points of a line of $B_k(m)$. To get a network, we will attach one terminal at each vertex of $B_k(m)$, so the property we need is that for any pairing of the vertices in $B_k(m)$ there exist edge-disjoint paths of $G_k(m)$ joining these pairs.

The definition of $G_k(m)$ is recursive. $G_1(m)$ is defined by adding a new vertex to $B_1(m)$ and join it to all of the m vertices of $B_1(m)$. Assume that $G_i(m)$ is defined for $i < k$. Then, we define $G_k(m)$ as follows:

(1) Add m^{k-1} distinct vertices to $B_k(m)$, one for each "horizontal" line, which is a set of points with the last $k - 1$ coordinates fixed. At each vertex include m edges going to every point of the corresponding line. The set of all vertices added is denoted by C.
(2) Let $B_1, \ldots, B_m$ be the "vertical" planes of $B_k(m)$, i.e., planes with the first coordinate fixed, each considered as a copy of $B_{k-1}(m)$. For each B_i, $i = 1, \ldots, m$, take six copies of $G_{k-1}(m)$ and identify with B_i their corresponding vertices belonging to $B_{k-1}(m)$.

Proposition 2. The graph $G_k(m)$ has

$$m^k + \frac{6^k - 1}{5} m^{k-1}$$

vertices, and a vertex x of $G_k(m)$ has the following degrees:

$$d(x) = \begin{cases} \dfrac{6^k - 1}{5} & \text{if } x \in B_k(m) \\ m & \text{if } x \notin B_k(m). \end{cases}$$

Proof. Simple counting shows that there are $(6^k - 1)m^{k-1}/5$ vertices of degree m added to $B_k(m)$ in steps (1) and (2). Since $G_k(m)$ is a bipartite graph and $|B_k(m)| = m^k$, the proposition follows. ■

Theorem 3. *$n(3, t) \leq 2t^{1+\varepsilon}$ holds for any $\varepsilon > 0$ and for sufficiently large t.*

Proof. Let k and m_0 be fixed integers such that $k \geq 2/\varepsilon$ and $m_0 \geq (6^k - 1)/5$. We show that $n(3, t) \leq 2t^{1+\varepsilon}$, for every $t \geq m_0^k$.

Assume for simplicity reasons that $t = m^k$, for some integer $m \geq m_0$ divisible by five. Let H be the network built on $G_k(m)$, by attaching m^k terminal nodes, one to each point of $B_k(m)$. We verify that for any pairing of the terminal nodes of H, i.e., that of the vertices of $B_k(m)$ in $G_k(m)$, there exist edge-disjoint paths of $G_k(m)$ joining the given pairs of vertices.

Partition the m vertical planes and thus the points of $B_k(m)$ into five equal classes A_i, $1 \leq i \leq 5$. Then, the pairs of the pairing of the points in $B_k(m)$ can be partitioned into 10 classes A_{ij}, $1 \leq i < j \leq 5$, such that if the pair xx' belongs to A_{ij}, then $\{x, x'\} \subset A_i \cup A_j$. For each class A_{ij}, we define a graph H_{ij} with m^{k-1} vertices, each corresponding to a horizontal line. The edges of H_{ij} are defined as follows: If a pair xx' of A_{ij} is such that x is on the horizontal line p and x' is on the horizontal line q, then the pair of vertices of H_{ij} corresponding to p and q define an edge of H_{ij}. Observe that H_{ij} may contain loops and multiple edges, and since $|A_i \cup A_j| = 2m/5$, its maximum degree is at most $2m/5$. Thus, by Shannon's theorem (see [12]), the chromatic index of H_{ij} is at most $(3/2)(2m)/5 = 3m/5$. Color the edges of H_{ij} with $3m/5$ colors in such a way that edges at the same vertex get distinct colors. For each color class, we may associate one of the $3m/5$ copies of B_i not in $A_i \cup A_j$. All the pairs defining loops and colored edges in H_{ij} can be realized by edge disjoint paths as follows: If xx' defines a loop, i.e., x and x' belong to the same horizontal line, then they can be joined by a path of length two including the point of C associated to that line. If x and x' are on different horizontal lines, then let B_l be the vertical plane associated to the color of xx'. Now starting from x and from x' one can reach the points x_1 and x_1' of B_l by paths of length two using distinct vertices of C (see [1]). The set of all pairs x_1, $x_1' \in B_l$ obtained from edges of H_{ij} and having the same color obviously forms a (partial) pairing of B_l. Thus, by induction, one can obtain edge disjoint paths realizing this pairing of B_l in one of the graphs $G_{k-1}(m)$ associated to B_l (see [2]). Since there are six copies of $G_{k-1}(m)$ on the same plane B_l, and a plane is used by at most six different A_{ij}'s, the 10 classes of pairs ($1 \leq i < j \leq 5$) can be handled simultaneously in distinct copies.

Since $m \geq m_0 \geq (6^k - 1)/5$, H has maximum degree m (see Proposition 2), and the argument above shows that H is an (m, m^k)-network. Using the substitution principle, each vertex can be replaced by a $(3, m)$-network containing at most m^2 interior nodes of degree three, as shown in Proposition 1. Thus, by the choice of k and m, we obtain

$$n(3, t) = n(3, m^k) \leq m^2 \left(m^k + \frac{6^k - 1}{5} m^{k-1}\right) \leq 2m^{k+2} = 2t^{1+2/k} \leq 2t^{1+\varepsilon}.$$

When t is not of the form m^k (or m is not divisible by five), the upper bound works out similarly. ■

6. COMPLETE NETWORKS

For certain Δ and t, (Δ, t)-networks built on complete (multi-)graphs tend to be optimal. The simplest case is when q terminal nodes are attached to each vertex of a K_p. Denote this network by K_p^q.

Proposition 3. *Assume that p is even. If K_p^q is a $(p + q - 1, pq)$-network, then $q \leq p/2$.*

Proof. Consider a pairing of K_p^q where the terminals hanging on a vertex of K_p are paired to terminals hanging on another vertex of K_p. Then, at most $p/2$ paths can be of length one among the edge disjoint paths realizing the pairing. Thus, $p/2 + 2((pq)/2 - p/2) \leq \binom{p}{2}$, giving $q \leq p/2$. ∎

The last proposition raises the question whether K_p^q is a $(p + q - 1, pq)$-network if $q = p/2$. A counting argument shows that this is not the case if $p \equiv 0 \pmod 4$, but it is probably true for $p \equiv 2 \pmod 4$.

Conjecture 2. *K_p^q is a $(p + q - 1, pq)$-network if $p \equiv 2 \pmod 4$ and $q = p/2$.*

The validity of the conjecture is checked in case of "clump-to-clump" pairing when the demand graph is a factor repeated $p/2$ times. Also, it is true if $p = 6$, and even this small example has some interesting features. The reader is challenged to find a realization for the demand graph in Figure 1. The best we can prove concerning Conjecture 2 is the following result:

Theorem 4. *K_p^q is a $(p + q - 1, pq)$-network if $q \leq p/(4 + 2\sqrt{3})$.*

Proof. Consider any pairing $x_i x_i'$, $i = 1, \ldots, pq/2$, of the terminals of K_p^q. Let D be the demand graph on the vertex set $V(K_p)$ associated to that pairing. Let $m(u, v)$ denote the multiplicity of an unordered pair (u, v) with $u, v \in V(D)$, $u \neq v$. Then, $M(D) = \Sigma_{(u,v)} (m(u, v) - 1)$ will be called the total multiplicity of D.

We describe an algorithm that transforms D into a graph G with no multiple edges [i.e., one with $M(G) = 0$] in several steps, by replacing the multiplied edges with paths of length two. Since $G \subset K_p^q$, we will obtain edge disjoint paths of K_p^q from the two-paths, the edges and the loops of G, by including the terminal edges. These edge-disjoint paths (of length four, three, and two) will represent the pairing indicated by the demand graph.

Our algorithm builds up a sequence of graphs $D_0 = D, D_1, \ldots, D_k = G$ in k stages, such that $M(D_k) = 0$, D_i has maximum degree $q + 2i$ or less, and $M(D_j) < M(D_i)$, for every $0 \leq i \leq k$ and $i < j \leq k$.

The ith stage of the algorithm transforms D_{i-1} into D_i in $p - 2q - 4i$ elementary steps, each step reducing the total multiplicity of G_{i-1} by one. When D_{i-1} progresses to D_i, the degree of a vertex is at most $q = 2i$. Hence, for every pair of distinct vertices, $u, v \in V(D)$, the set $S_i(u, v)$ of all vertices nonadjacent to both u and v has cardinality more than $p - 2(q + 2i)$. Assuming that there are multiple edges between u and v, an elementary step consists of picking a vertex z from $S_i(u, v)$ then including the edges of the two-path (u, z, v) into the graph and removing one edge between u and v.

As long as there exist multiple edges in D_{i-1}, we repeat the elementary steps on arbitrary pairs u, v, picking distinct midpoints for the two paths. With the choice of distinct vertices in the same stage, we avoid the multiplication of the edges introduced earlier; moreover, we ensure that the degree of each vertex increases by at most two. Clearly, an elementary step can be done at least $(p - 2q - 4i)$-times; hence, at the end of the ith stage, $M(D_i) \leq M(D_{i-1}) - (p - 2q - 4i)$. Notice that $i \leq (p - 2q)/4$ is sufficient to reduce $M(D_{i-1})$ by at least one.

Observing that $M(D) \leq p(q - 1)/2$, easy arithmetic shows that the total multiplicity of the demand graph reduces to zero in at most $k = (p - 2q)/4$ stages, i.e.,

$$\sum_{i=0}^{k} (p - 2q - 4i) \leq \frac{p}{2}(q - 1)$$

holds, whenever $q \leq p/(4 + 2\sqrt{3})$. ■

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