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Graphs with k odd cycle lengths

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Abstract

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If G is a graph with $k \geqslant 1$ odd cycle lengths then each block of G is either K_{2k+2} or contains a vertex of degree at most $2k$. As a consequence, the chromatic number of G is at most $2k+2$.

For a graph G let $L(G)$ denote the set of odd cycle lengths of G, i.e.,

$$L(G) = \{2i+1\colon G \text{ contains a cycle of length } 2i+1\}.$$

With this notation, bipartite graphs are the graphs with $|L(G)| = 0$. Bollobás and Erdős asked how large can the chromatic number of G be if $|L(G)| = k$. They conjectured that $|L(G)| = k$ implies $\chi(G) \leqslant 2k+2$ and this is best possible considering $G = K_{2k+2}$.

The case $k = 1$ is checked by Bollobás and Shelah (see [1, p. 472] for the motivation). Gallai suspected that a stronger statement is true, namely if G is 2-connected, $|L(G)| = k$, $G \neq K_{2k+2}$ then the minimum degree of G is at most $2k$. The aim of this paper is to prove this stronger version of the original conjecture.

Theorem 1. *If G is a 2-connected graph with minimum degree at least $2k+1$ then $|L(G)| = k \geqslant 1$ implies $G = K_{2k+2}$.*

Assuming that $|L(G)| = k$, Theorem 1 clearly allows to color the vertices of the blocks of G with at most $2k+1$ colors except when a block is a K_{2k+2}. Thus the following corollary is obtained.

Corollary. *If $|L(G)| = k \geqslant 1$ then the chromatic number of G is at most $2k+1$, unless some block of G is a K_{2k+2}. (If there is such a block, then the chromatic number of G is $2k+2$.)*

For the proof of Theorem 1 and for the lemmas we adopt the following notation: Graph G is a 2-connected graph with minimum degree at least $2k+1$ and with $|L(G)|=k\geq 1$. Let C denote a longest odd cycle of G. The subgraph of G induced by $V(G)-V(C)$ is denoted by G'. A longest path of G' is denoted by S, A and B are the endpoints of S. $|C|$ and $|S|$ denote the length of C and S. If S is a path and x, y are two vertices of S then $S(x, y)$ denotes the subpath of S between x and y. $\Gamma(x)$ denotes the set of vertices adjacent with x in G. The degree of a vertex is denoted by $d(x)$.

Proof of Theorem 1. If $A=B$, i.e., G' has no edges then Lemma 5 implies $G=K_{2k+2}$. We may assume therefore that $A\neq B$, i.e., $|S|\geq 1$.

If $\Gamma(A)\cap C=\emptyset$ (or $\Gamma(B)\cap C=\emptyset$) then $d(A)\geq 2k+1$ (or $d(B)\geq 2k+1$) implies that G' contains a cycle with $2k-1$ diagonals incident to the same vertex (A or B) of the cycle. Let H be this subgraph of G'. Applying Lemma 2, either $|L(H)|\geq k$ or H is bipartite. The former case leads to a contradiction because of Lemma 1. Therefore H is bipartite. Since G is 2-connected, there exist two vertex-disjoint paths S_1 and S_2 joining $V(C)$ and $V(H)$. Apply Lemma 3 with $x=V(H)\cap S_1$ and $y=V(H)\cap S_2$. The $k+1$ paths ensured by Lemma 3 together with S_1 and S_2 and with the arc of C of suitable parity define $k+1$ odd cycles of different lengths. Thus we get a contradiction again.

We conclude that $\Gamma(A)\cap C\neq\emptyset$ and $\Gamma(B)\cap C\neq\emptyset$. Due to the symmetry of A and B we may assume that

$$1\leq p=|\Gamma(A)\cap C|\leq|\Gamma(B)\cap C|.$$

Let $|\Gamma(A)\cap S|=q+1$, that is there are q diagonals of S starting from A. Since $d(A)\geq 2k+1$, $p+q\geq 2k$ follows.

Case 1: $\Gamma(A)\cap C\neq\Gamma(B)\cap C$.

Lemma 4 implies

$$|L(G)|\geq\left\lceil\frac{p}{2}\right\rceil+q\geq\left\lceil\frac{2k-q}{2}\right\rceil+q=k-\left\lfloor\frac{q}{2}\right\rfloor+q>k,$$

leading to a contradiction unless $q=0$ and $p+q=2k$, i.e., $p=2k$. This case is handled in Lemma 8 and also leads to $|L(G)|\geq k+1$.

Case 2: $\Gamma(A)\cap C=\Gamma(B)\cap C$.

Now lemma 4 can be applied with $p-1$ in the role of p and we get

$$|L(G)|\geq\left\lceil\frac{p-1}{2}\right\rceil+q\geq\left\lceil\frac{2k-q-1}{2}\right\rceil+q=k-\left\lfloor\frac{q+1}{2}\right\rfloor+q>k$$

unless $q=1$ and $p+q=2k$, or $q=0$ and $p+q$ is $2k$ or $2k+1$. The case $q=1$ is treated in Lemma 6 and the case $q=0$ is treated in Lemma 7. In both cases we get $k+1$ odd cycles of different lengths leading to a contradiction. Thus the only possibility is that $G=K_{2k+2}$ and Theorem 1 is proved. □

Lemma 1. *If C' is an odd cycle of G' then $|C'|<|C|$.*

Proof. Assume that $|C'| \geq |C|$. Since G is 2-connected, there exist two vertex-disjoint paths P_1 and P_2 in G such that $|P_i \cap C| = 1$ and $|P_i \cap C'| = 1$ for $i = 1, 2$. The subgraph $C \cup C' \cup P_1 \cup P_2$ is clearly the union of two odd cycles C_1 and C_2. Since $|C_1| + |C_2| > |C| + |C'| \geq 2\,|C|$, $|C_1|$ or $|C_2|$ is larger than $|C|$ and we reach a contradiction. □

Lemma 2. *Let H be a graph we get from a cycle T by adding $2k-1$ diagonals, each of them incident to the same vertex x_0 of T. Then either H is a bipartite graph or $|L(H)| \geq k$.*

Proof. Assume that T has diagonals $e_i = x_0x_i$ for $i = 1, 2, \ldots, x_{2k-1}$. W.l.o.g., $x_0, x_1, \ldots, x_{2k-1}$ follow each other on T in this order. Let a_i denote the length of the path on T from x_{i-1} to x_i following the order of $x_j - s$. Consider x_0 as x_{2k} for convenience. The proof is by induction on k. The case $k = 1$ is clear. Assume that the lemma is true for each $k' < k$ and $k \geq 2$. If T is odd then each e_i divides $T \cup e_i$ into an odd and an even cycle. Let C_i denote the odd cycle for $i = 1, 2, \ldots, 2k - 1$. If e and f are the edges of T incident to x_0 then each C_i contains either e or f. There are k indices i such that C_i contains one of $\{e, f\}$. These cycles satisfy the requirements of the lemma.

If T is an even cycle, e_i is called even if it divides T into two even cycles, otherwise it is called odd. If the only odd diagonal of T is e_k then we can define k odd cycles as follows. Let C_i be the cycle of H containing e_i and e_k for $i = 1, 2, \ldots, k-1$, and let C_k be the cycle containing e_k and the path on T from x_0 to x_k containing $x_1, \ldots, x_{k-1}$. Otherwise let p be the smallest integer in $\{1, 2, \ldots, k-1\}$ such that either e_p or e_{2k-p} is an odd diagonal. If no such p exists then H is bipartite. Apply the inductive hypothesis to H^* defined as the cycle C^* determined by e_p and e_{2k-p} in T together with the diagonals e_i for $p < i < 2k - p$. If H^* is bipartite then both e_p and e_{2k-p} are odd diagonals of T. By symmetry, one may assume that $a_p \geq a_{2k-p+1}$. Let C_i be the cycle determined by the diagonals e_i and e_{2k-p+1} for $i = p, \ldots, 2k-p$. Let D_i be the cycle determined by the diagonals e_i and e_{2k-p} for $i = 1, 2, \ldots, p-1$. Let D_0 be the cycle determined by e_{2k-p} and the path on T from x_0 to x_{2k-p} following the order $x_0, x_1, \ldots, x_{2k-p}$. All these cycles are odd and $a_p \geq a_{2k-p+1}$ ensures that

$$|C_{2k-p}| < \cdots < |C_p| \leq |D_{p-1}| < \cdots < |D_1| < |D_0|.$$

Therefore we defined at least $2k - 2p + 1 + p - 1 = 2k - p$ odd cycles of different lengths. Clearly $2k - p > k$ since $k \geq p + 1$ by the definition of p.

If H^* is not bipartite then by induction it contains odd cycles $C_1, C_2, \ldots, C_{k-p}$ such that $|C_1| < |C_2| < \cdots < |C_{k-p}|$. Let D_i be the cycle determined by e_p and e_{2k-p+i} for $i = 1, 2, \ldots, p-1$. Let D_p be the cycle determined by e_p and the path $x_0, x_1, \ldots, x_p$ on T. Clearly $|C_{k-p}| < |D_1| < \cdots < |D_p|$ thus we have the desired cycles. □

Lemma 3. *Let H be a graph we get from a cycle T by adding $2k-1$ diagonals $e_i = x_0x_i$ to T. Assume that H is a bipartite graph and $x, y \in V(H)$, $x \neq y$. Then there exist $k+1$ paths $P_1, \ldots, P_{k+1}$ in H from x to y such that $|P_i| \equiv |P_j| \bmod 2$ and $|P_i| \neq |P_j|$ for $1 \leq i < j \leq k+1$.*

Proof. If $x_0 = x$ or $x_0 = y$, say $x_0 = x$ then one of the two xy paths of T contains k vertices of $X = \{x_1, \ldots, x_{2k-1}\}$, say $x_1, x_2, \ldots, x_k$. The paths $x_0e_ix_i \cdots x_ky$ for $i = 1, 2, \ldots, k$ and the path $x_0x_1x_2 \cdots x_ky$ satisfy the requirements.

If $x_0 \notin \{x, y\}$, assume that one of the paths from x to y in T contains $x_1, x_2, \ldots, x_m, x_0, x_{n+1}, x_{n+2}, \ldots, x_{2k-1}$ in this order and the other xy path on T contains vertices $x_{m+1}, x_{m+2}, \ldots, x_n$ in this order. Then $xx_ie_ix_0e_jx_j \cdots x_ny$ for $i = 1, 2, \ldots, m$, $j = m+1, \ldots, n$ obviously contains n paths of the same parity and of different lengths. Similarly, $xx_ie_ix_0e_jx_j \cdots x_{2k-1}y$ for $i = 1, 2, \ldots, m$, $j = n+1, \ldots, 2k-1$ and $xx_1x_2 \cdots x_mx_0x_{n+1} \cdots x_{2k-1}$ determines $m + 2k - n$ paths of the same parity and of different lengths. The two path systems have $m + 2k$ paths together therefore one of them have at least $k+1$ paths unless $m = 0$. But in this case $n = k$ and we can apply the same argument by exchanging the role of x and y. □

Lemma 4. *Assume that A is adjacent to p vertices, $y_1, y_2, \ldots, y_p$ of C and to $q+1$ vertices of S. Moreover, B is adjacent to $y \in V(C) - \{y_1, \ldots, y_p\}$. Then $|L(G)| \geq \lceil p/2 \rceil + q$.*

Proof. Assume that $y_1, y_2, \ldots, y_p$ and y follow each other in this order along C. Let $Az_1, Az_2, \ldots, Az_q$ be ghe diagonals of S in this order starting from A. We shall define $q+1$ paths S_i and p paths P_i as follows:

$$S_1 = S, \qquad S_{i+1} = A, z_i, z_{i+1}, \ldots, z_q, B \quad \text{for } i = 1, 2, \ldots, q$$
$$P_i = A, y_i, y_{i+1}, \ldots, y_p, y, B \quad \text{for } i = 1, 2, \ldots, p.$$

Let a denote the length of the subpath z_qB on S. It is easy to check that the following numbers are all odd cycle lengths:

$$f_i = |S_i| - a + 1 \quad \text{if } |S_i| - a \text{ is even,} \tag{1}$$
$$g_{ij} = |S_i| + |P_j| \quad \text{if } |S_i| + |P_j| \text{ is odd,} \tag{2}$$
$$h_{ij} = |S_i| + |C| - |P_j| + 2 \quad \text{if } |S_i| + |P_j| \text{ is even.} \tag{3}$$

By the symmetry of (2) and (3) one may assume that a is even.

Let i, j, m be indices such that

$$|S_i| \text{ is even,} \qquad |S_j| \text{ is odd,} \qquad |P_m| \text{ is even.} \tag{4}$$

If $f_i \geq g_{jm}$, i.e., $|S_i| - a + 1 \geq |S_j| + |P_m|$ then (3) implies that

$$h_{im} = |S_i| + |C| - |P_m| + 2 \geq |S_j| + |P_m| + a - 1 + |C| - |P_m| + 2$$
$$= |S_j| + 1 + a + |C| > |C|$$

contradicting the maximality of C. Therefore $f_i < g_{jm}$ for each choice of indices satisfying (4). Thus g_{jm} takes at least $J + M - 1$ values where

$$J = |\{j\colon |S_j| \text{ is odd}\}|, \qquad M = |\{m\colon |P_m| \text{ is even}\}|.$$

If $I = |\{i\colon |S_i| \text{ is even}\}|$ then $f_i < g_{jm}$ implies that f_i and g_{jm} together take at least $q + M$ values, i.e.,

$$|L(G)| \geqslant q + M. \tag{5}$$

Similarly, let i, j, n be indices such that

$$|S_i| \text{ is even}, \qquad |S_j| \text{ is odd}, \qquad |P_n| \text{ is odd}. \tag{6}$$

If $f_i \geqslant h_{jn}$, i.e., $|S_i| - a + 1 \geqslant |S_j| + |C| - |P_n| + 2$ then (2) implies

$$\begin{aligned} g_{in} = |S_i| + |P_n| &\geqslant |S_j| + |C| - |P_n| + 2 + a - 1 + |P_n| \\ &= |S_j| + a + 1 + |C| > |C| \end{aligned}$$

leading to a contradiction. Therefore $f_i < h_{jn}$ for each choice of indices satisfying (6). Thus for $N = |\{n\colon |P_n| \text{ is odd}\}|$ we get

$$|L(G)| \geqslant q + N. \tag{7}$$

Since $M + N = p$, either M or N is at least $\lceil p/2 \rceil$ and the lemma follows from (5) and (7). □

Lemma 5. *If $V(G) - V(C)$ is an independent set then $G = K_{2k+2}$.*

Proof. If $V(G) - V(C) = \emptyset$ then $|L(G) \geqslant k + 1$ follows from Lemma 2 and we have a contradiction. Select a vertex $T \in V(G) - V(C)$. Then $d(T) \geqslant 2k + 1$ implies that T is adjacent to $X = \{x_1, x_2, \ldots, x_{2k+1}\} \subseteq V(C)$ and assume that the vertices of X follow each other in this order. This order gives an orientation to C. Set $A_i^x = a_1^x + a_2^x + \cdots + a_i^x$ where $x = x_m$ and a_j^x is the length of the path on C from x_{m+j-1} to x_{m+j}, $i = 1, 2, \ldots, 2k + 1$. Here indices are taken modulo $2k + 1$ and the path is understood according the fixed orientation of C. Clearly $A_{2k+1}^x = |C|$ for any $x \in X$. Moreover, for any $x \in X$ we have the following odd cycle lengths:

$$\begin{array}{ll} A_i^x + 2 & \text{if } A_i^x \text{ is odd and } 1 \leqslant i < 2k + 1, \\ |C| - A_i^x + 2 & \text{if } A_i^x \text{ is even and } 1 \leqslant i \leqslant 2k + 1. \end{array} \tag{8}$$

For any $x \in X$ there are $k + 1$ elements of $\{A_i^x\}_{i=1}^{2k+1}$ having the same parity. If they are even then (8) implies that $|L(G)| \geqslant k + 1$, a contradiction. Thus we may assume that for each $x \in X$, $\{A_i^x\}_{i=1}^{2k+1}$ contains exactly $k + 1$ odd numbers and $A_i^x + 2 = |C|$ for some i. This implies that X divides C into paths of length 1 or 2. If $A_1^x = 2$ for some $x \in X$ then there are k odd numbers in $\{A_i^x\}_{i=1}^{2k}$ but all of them are larger than 3. Since $|C|$ is odd, there exists $y \in X$, $A_1^y = 1$ and we have a 3-cycle giving $k + 1$ odd cycle lengths altogether. Thus the only possibility is that

X divides C into paths of length 1, i.e., $|C| = 2k+1$. Now $V(G) - V(C) = \{T\}$ otherwise we get a cycle of length $2k+3$, thus G has odd cycle lengths $2i+1$ for $i = 1, 2, \ldots, k+1$, leading to a contradiction. Thus $|V(G)| = 2k+2$ and since each degree of G is at least $2k+1$, $G = K_{2k+2}$. □

Lemma 6. *If S has one diagonal at A and one at B, and, moreover, $\Gamma(A) \cap C = \Gamma(B) \cap C$, $|\Gamma(A) \cap C| = 2k-1$ then $|L(G)|) \geq k+1$.*

Proof. Let $\Gamma(A) \cap C = \Gamma(B) \cap C = X = \{x_1, x_2, \ldots, x_{2k-1}\}$ and assume that $x_1, x_2, \ldots, x_{2k-1}$ is their order on C. We use the notation A_i^x for $i = 1, 2, \ldots, 2k-1$ as defined in Lemma 5. First we assume that $k \geq 2$.

Let W denote the graph consisting of S with its two diagonals. Let $R(W)$ denote the set of path lengths in W from A to B. It is easy to check that $|R(W)| \geq 3$ except when $R(W) = \{b+2, 3b+2\}$ or when $R(W) = \{2, b+2\}$ for some b (in these cases S has two crossing diagonals and b is the length of the middle segment of S).

If $x, y \in X$, $x \neq y$, then x and y can be connected by paths of length 2 and $h+2$ for $h \in R(W)$. Therefore we have the following odd cycle lengths:

$$\begin{aligned}
&A_i^x + 2,\ A_i^x + h + 2 \quad \text{if } A_i^x \text{ is odd and } h \in R(W) \text{ is even,}\\
&A_i^x + h + 2 \quad \text{if } A_i^x \text{ is even and } h \in R(W) \text{ is odd,}\\
&|C| - A_i^x + 2,\ |C| - A_i^x + h + 2 \quad \text{if } A_i^x \text{ is even and } h \in R(W) \text{ is even,}\\
&|C| - A_i^x + h + 2 \quad \text{if } A_i^x \text{ is odd and } h \in R(W) \text{ is odd.}
\end{aligned} \tag{9}$$

The index i in (9) can take values $1, 2, \ldots, 2k-2$.

If there are at least k A_i^x of the same parity for some $x \in X$ then selecting two numbers from $\{2\} \cup \{h+2: h \in R(W)\}$ of the same parity, (9) implies $|L(G)| \geq k+1$, contradiction. Assume that A_1^x is even for some $x \in X$. Then $A_1^y, \ldots, A_{2k-3}^y$ contain $k-1$ numbers of the same parity where y is the vertex in X following x. Adding A_1^x to the largest of these numbers we get k different numbers of the same parity. Therefore A_1^x is odd for each $x \in X$. Consequently

$$\min\{A_i^x : x \in X\} = \min\{A_1^x : x \in X\}$$

is an odd number thus $\min\{A_1^x : x \in X\} + 2$ and $A_i^x + h + 2$ gives $k+1$ odd numbers if A_i^x is even and h is odd, provided that $R(W)$ has two odd numbers. If $|R(W)| \geq 3$ then this is true because $\{2\} \cup R(W)$ must contain two odd and two even numbers (three of the same parity would give $k+1$ odd cycles by the second or fourth line of (9)). But if $|R(W)| = 2$ then W is described before. If $R(W) = \{b+2, 3b+2\}$ then for even b, $\{2\} \cup R(W)$ contains three even numbers, for odd b, $R(W)$ contains two odd numbers. If $R(W) = \{2, b+2\}$ then for even b, $\{2\} \cup R(W)$ contains three even numbers. For odd b one can find $k+1$

odd numbers as follows:

$$\min\{A_1^x : x \in X\} + 2, \qquad \min\{A_1^x : x \in X\} + 4,$$

$A_i^x + b + 4$ for A_i^x even.

Finally, the case $k = 1$ is treated. Now $|X| = 1$ and using the 2-connectedness of G, there is a path from $V(C) - X$ to S. One can easily find two odd cycles of different lengths, the missing details are omitted here. □

Lemma 7. *If $\Gamma(A) \cap C = \Gamma(B) \cap C$ and $|\Gamma(A) \cap C| = 2k$, $A \neq B$ then $|L(G)| \geqslant k + 1$.*

Proof. Assume that $\Gamma(A) \cap C = X = \{x_1, x_2, \ldots, x_{2k}\}$ and we shall use the notations of the previous lemmas. Let s denote the length of S. It is clear that for each $x \in X$ $A_1^x, A_2^x, \ldots, A_{2k-1}^x$ must consist of $k - 1$ numbers of one parity and k numbers of the other parity, otherwise $|L(G)| \geqslant k + 1$ follows. It is easy to check that if $\{A_i^x\}_{i=1}^{2k-1}$ contains $k - 1$ odd numbers then $\{A_i^y\}_{i=1}^{2k-1}$ contains k odd numbers for $y = x_{j+1}$ if $x = x_j$.

If s is odd then select x such that $\{A_i^x\}_{i=1}^{2k-1}$ contains $k - 1$ odd numbers, i.e., k even numbers: $b_1 < b_2 < \cdots < b_k$. Then we have $k + 1$ different odd cycle lengths: $s + 2$, $b_i + s + 2$ for $i = 1, \ldots, k$.

If s is even then select x such that A_i^x contains k odd numbers: $b_1 < b_2 < \cdots < b_k$. Now $b_1 + 2 < b_2 + 2 < \cdots < b_k + 2 < b_k + 2 + s$ are different odd cycle lengths. □

Lemma 8. *If $|\Gamma(A)| \cap C| = 2k$ and $y \in (\Gamma(B) \cap C) - \Gamma(A)$ then $|L(G)| \geqslant k + 1$.*

Proof. Assume that $\Gamma(A) \cap C = \{x_1, x_2, \ldots, x_{2k}\} = X$ and $y = x_0, x_1, \ldots, x_{2k}$ follow each other on C in this order. Let a_i be the length of the path connecting x_i and x_{i-1} on C which does not contain other x_j. The length of S is denoted by s. Set $A_i = a_1 + \cdots + a_i$ for $i = 1, 2, \ldots, 2k$. We may assume that $\{A_i\}_{i=1}^{2k}$ contains k odd and k even numbers otherwise $|L(G)| \geqslant k + 1$ is obvious.

Case 1: $a_1 + s \equiv 0 \bmod 2$.

Let I be the set of those indices for which $A_i + s + 2$ is odd. Clearly, $|I| = k$ and for $i, j \in I$, $A_i - a_1 + 2 \equiv A_j + s + 2$, therefore $A_i - a_1 + 2$ and $A_j + s + 2$ are odd cycle lengths in G. Let A_j be the smallest element of $\{A_i : i \in I\}$, then $A_j - a_1 + 2 < A_i + s + 2$ for $i \in I$ since $A_j < A_i + s + a_1$ for $i \in I$. Therefore A_j and $A_i + s + 2$ for $i \in I$ gives $k + 1$ different odd cycle lengths.

Case 2: $a_1 + s \equiv 1 \bmod 2$.

In this case $a_1 + s + 2$ is an odd cycle length. Let I be the set of indices i such that $|C| - A_i + a_1 + 2$ is odd. Clearly, $|I| = k$. We claim that $a_1 + s + 2 \neq |C| - A_i + a_1 + 2$ for $i \in I$. If there is equality for some $i \in I$ then $A_i = |C| - s$. Since

$|C| - A_i + a_1 + 2$ is odd, $A_i \equiv a_1 \bmod 2$ therefore $A_i + 2 + s \equiv a_1 + 2 + s \equiv 1 \bmod 2$. Thus $A_i + 2 + s = |C| - s + 2 + s = |C| + 2$ and $A_i + 2 + s$ is the length of an odd cycle. This contradicts the maximality of C. Thus $a_1 + s + 2$ is different from $|C| - A_i + a_1 + 2$ for $i \in I$ and we have $k + 1$ different odd cycle lengths.

Reference

[1] P. Erdős, Some of my favourite unsolved problems, in: A. Baker, B. Bollobás and A. Hajnal, eds., A tribute to Paul Erdős (Cambridge Univ. Press, Cambridge, 1990) 467.