Graphs with $k$ odd cycle lengths

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Abstract


If $G$ is a graph with $k \geq 1$ odd cycle lengths then each block of $G$ is either $K_{2k+2}$ or contains a vertex of degree at most $2k$. As a consequence, the chromatic number of $G$ is at most $2k + 2$.

For a graph $G$ let $L(G)$ denote the set of odd cycle lengths of $G$, i.e.,

$$L(G) = \{2i + 1: G \text{ contains a cycle of length } 2i + 1\}.$$ 

With this notation, bipartite graphs are the graphs with $|L(G)| = 0$. Bollobás and Erdős asked how large can the chromatic number of $G$ be if $|L(G)| = k$. They conjectured that $|L(G)| = k$ implies $\chi(G) \leq 2k + 2$ and this is best possible considering $G = K_{2k+2}$.

The case $k = 1$ is checked by Bollobás and Shelah (see [1, p. 472] for the motivation). Gallai suspected that a stronger statement is true, namely if $G$ is 2-connected, $|L(G)| = k, G \neq K_{2k+2}$ then the minimum degree of $G$ is at most $2k$. The aim of this paper is to prove this stronger version of the original conjecture.

**Theorem 1.** If $G$ is a 2-connected graph with minimum degree at least $2k + 1$ then $|L(G)| = k \geq 1$ implies $G = K_{2k+2}$.

Assuming that $|L(G)| = k$, Theorem 1 clearly allows to color the vertices of the blocks of $G$ with at most $2k + 1$ colors except when a block is a $K_{2k+2}$. Thus the following corollary is obtained.

**Corollary.** If $|L(G)| = k \geq 1$ then the chromatic number of $G$ is at most $2k + 1$, unless some block of $G$ is a $K_{2k+2}$. (If there is such a block, then the chromatic number of $G$ is $2k + 2$.)
For the proof of Theorem 1 and for the lemmas we adopt the following notation: Graph \( G \) is a 2-connected graph with minimum degree at least \( 2k + 1 \) and with \( |L(G)| = k \geq 1 \). Let \( C \) denote a longest odd cycle of \( G \). The subgraph of \( G \) induced by \( V(G) - V(C) \) is denoted by \( G' \). A longest path of \( G' \) is denoted by \( S \). Let \( A \) and \( B \) be the endpoints of \( S \). If \( S \) is a path and \( x, y \) are two vertices of \( S \) then \( S(x, y) \) denotes the subpath of \( S \) between \( x \) and \( y \). \( T(x) \) denotes the set of vertices adjacent with \( x \) in \( G \). The degree of a vertex is denoted by \( d(x) \).

**Proof of Theorem 1.** If \( A = B \), i.e., \( G' \) has no edges then Lemma 5 implies \( G = K_{2k+2} \). We may assume therefore that \( A \neq B \), i.e., \( |S| \geq 1 \).

If \( \Gamma(A) \cap C = \emptyset \) (or \( \Gamma(B) \cap C = \emptyset \)) then \( d(A) \geq 2k + 1 \) (or \( d(B) \geq 2k + 1 \)) implies that \( G' \) contains a cycle with \( 2k - 1 \) diagonals incident to the same vertex \( A \) or \( B \) of the cycle. Let \( H \) be this subgraph of \( G' \). Applying Lemma 2, either \( |L(H)| \geq k \) or \( H \) is bipartite. The former case leads to a contradiction because of Lemma 1. Therefore \( H \) is bipartite. Since \( G \) is 2-connected, there exist two vertex-disjoint paths \( S_1 \) and \( S_2 \) joining \( V(C) \) and \( V(H) \). Applying Lemma 3 with \( x = V(H) \cap S_1 \) and \( y = V(H) \cap S_2 \). The \( k + 1 \) paths ensured by Lemma 3 together with \( S_1 \) and \( S_2 \) and with the arc of \( C \) of suitable parity define \( k + 1 \) odd cycles of different lengths. Thus we get a contradiction again.

We conclude that \( \Gamma(A) \cap C \neq \emptyset \) and \( \Gamma(B) \cap C \neq \emptyset \). Due to the symmetry of \( A \) and \( B \) we may assume that

\[
1 \leq p = |\Gamma(A) \cap C| \leq |\Gamma(B) \cap C|.
\]

Let \( |\Gamma(A) \cap S| = q + 1 \), that is there are \( q \) diagonals of \( S \) starting from \( A \). Since \( d(A) \geq 2k + 1 \), \( p + q \geq 2k \) follows.

**Case 1:** \( \Gamma(A) \cap C \neq \Gamma(B) \cap C \).

Lemma 4 implies

\[
|L(G)| \geq \left\lceil \frac{p}{2} \right\rceil + q \geq \left\lceil \frac{2k - q}{2} \right\rceil + q = k - \left\lfloor \frac{q}{2} \right\rfloor + q > k,
\]

leading to a contradiction unless \( q = 0 \) and \( p + q = 2k \), i.e., \( p = 2k \). This case is handled in Lemma 8 and also leads to \( |L(G)| \geq k + 1 \).

**Case 2:** \( \Gamma(A) \cap C = \Gamma(B) \cap C \).

Now lemma 4 can be applied with \( p - 1 \) in the role of \( p \) and we get

\[
|L(G)| \geq \left\lceil \frac{p - 1}{2} \right\rceil + q \geq \left\lceil \frac{2k - q - 1}{2} \right\rceil + q = k - \left\lfloor \frac{q + 1}{2} \right\rfloor + q > k
\]

unless \( q = 1 \) and \( p + q = 2k \), or \( q = 0 \) and \( p + q = 2k \) or \( 2k + 1 \). The case \( q = 1 \) is treated in Lemma 6 and the case \( q = 0 \) is treated in Lemma 7. In both cases we get \( k + 1 \) odd cycles of different lengths leading to a contradiction. Thus the only possibility is that \( G = K_{2k+2} \) and Theorem 1 is proved. \( \square \)

**Lemma 1.** If \( C' \) is an odd cycle of \( G' \) then \( |C'| < |C| \).
Proof. Assume that $|C'| \geq |C|$. Since $G$ is 2-connected, there exist two vertex-disjoint paths $P_1$ and $P_2$ in $G$ such that $|P_1 \cap C| = 1$ and $|P_2 \cap C'| = 1$ for $i = 1, 2$. The subgraph $C \cup C' \cup P_1 \cup P_2$ is clearly the union of two odd cycles $C_1$ and $C_2$. Since $|C_1| + |C_2| > |C| + |C'| \geq 2|C|$, $|C_1|$ or $|C_2|$ is larger than $|C|$ and we reach a contradiction. \( \square \)

Lemma 2. Let $H$ be a graph we get from a cycle $T$ by adding $2k - 1$ diagonals, each of them incident to the same vertex $x_0$ of $T$. Then either $H$ is a bipartite graph or $|L(H)| \geq k$.

Proof. Assume that $T$ has diagonals $e_i = x_0 x_i$ for $i = 1, 2, \ldots, x_{2k-1}$. W.l.o.g., $x_0, x_1, \ldots, x_{2k-1}$ follow each other on $T$ in this order. Let $a_i$ denote the length of the path on $T$ from $x_{i-1}$ to $x_i$ following the order of $x_j - s$. Consider $x_0$ as $x_{2k}$ for convenience. The proof is by induction on $k$. The case $k = 1$ is clear. Assume that the lemma is true for each $k' < k$ and $k \geq 2$. If $T$ is odd then each $e_i$ divides $T \cup e_i$ into an odd and an even cycle. Let $C_i$ denote the odd cycle for $i = 1, 2, \ldots, 2k - 1$. If $e$ and $f$ are the edges of $T$ incident to $x_0$ then each $C_i$ contains either $e$ or $f$. There are $k$ indices $i$ such that $C_i$ contains one of $\{e, f\}$. These cycles satisfy the requirements of the lemma.

If $T$ is an even cycle, $e_i$ is called even if it divides $T$ into two even cycles, otherwise it is called odd. If the only odd diagonal of $T$ is $e_k$ then we can define $k$ odd cycles as follows. Let $C_i$ be the cycle of $H$ containing $e_i$ and $e_k$ for $i = 1, 2, \ldots, k - 1$, and let $C_k$ be the cycle containing $e_k$ and the path on $T$ from $x_0$ to $x_k$ containing $x_1, \ldots, x_{k-1}$. Otherwise let $p$ be the smallest integer in $\{1, 2, \ldots, k - 1\}$ such that either $e_p$ or $e_{2k-p}$ is an odd diagonal. If no such $p$ exists then $H$ is bipartite. Apply the inductive hypothesis to $H^*$ defined as the cycle $C^*$ determined by $e_p$ and $e_{2k-p}$ in $T$ together with the diagonals $e_i$ for $p < i < 2k - p$. If $H^*$ is bipartite then both $e_p$ and $e_{2k-p}$ are odd diagonals of $T$. By symmetry, one may assume that $a_p \geq a_{2k-p+1}$. Let $C_i$ be the cycle determined by the diagonals $e_i$ and $e_{2k-p+1}$ for $i = p, \ldots, 2k - p$. Let $D_i$ be the cycle determined by the diagonals $e_i$ and $e_{2k-p}$ for $i = 1, 2, \ldots, p - 1$. Let $D_0$ be the cycle determined by $e_{2k-p}$ and the path on $T$ from $x_0$ to $x_{2k-p}$ following the order $x_0, x_1, \ldots, x_{2k-p}$. All these cycles are odd and $a_p \geq a_{2k-p+1}$ ensures that

$$|C_{2k-p}| < \cdots < |C_p| \leq |D_{p-1}| < \cdots < |D_1| < |D_0|.$$  

Therefore we defined at least $2k - 2p + 1 + p - 1 = 2k - p$ odd cycles of different lengths. Clearly $2k - p > k$ since $k \geq p + 1$ by the definition of $p$.

If $H^*$ is not bipartite then by induction it contains odd cycles $C_1, C_2, \ldots, C_{k-p}$ such that $|C_1| < |C_2| < \cdots < |C_{k-p}|$. Let $D_i$ be the cycle determined by $e_p$ and $e_{2k-p+i}$ for $i = 1, 2, \ldots, p - 1$. Let $D_p$ be the cycle determined by $e_p$ and the path $x_0, x_1, \ldots, x_p$ on $T$. Clearly $|C_{k-p}| < |D_1| < \cdots < |D_p|$ thus we have the desired cycles. \( \square \)
Lemma 3. Let $H$ be a graph we get from a cycle $T$ by adding $2k - 1$ diagonals $e_i = x_0x_i$ to $T$. Assume that $H$ is a bipartite graph and $x, y \in V(H)$, $x \neq y$. Then there exist $k + 1$ paths $P_1, \ldots, P_{k+1}$ in $H$ from $x$ to $y$ such that $|P_i| \equiv |P_j| \mod 2$ and $|P_i| \neq |P_j|$ for $1 \leq i < j \leq k + 1$.

Proof. If $x_0 = x$ or $y_0 = y$, say $x_0 = x$ then one of the two $xy$ paths of $T$ contains $k$ vertices of $X = \{x_1, \ldots, x_{2k-1}\}$, say $x_1, x_2, \ldots, x_k$. The paths $x_0 e_i x_1 \cdots x_k y$ for $i = 1, 2, \ldots, k$ and the path $x_0 x_1 x_2 \cdots x_k y$ satisfy the requirements.

If $x_0 \notin \{x, y\}$, assume that one of the paths from $x$ to $y$ in $T$ contains $x_1, x_2, \ldots, x_m, x_0, x_{n+1}, x_{n+2}, \ldots, x_{2k-1}$ in this order and the other $xy$ path on $T$ contains vertices $x_{m+1}, x_{m+2}, \ldots, x_n$ in this order. Then $xx_i e_i x_0 e_j x_1 \cdots x_k y$ for $i = 1, 2, \ldots, m, j = m + 1, \ldots, n$ obviously contains $n$ paths of the same parity and of different lengths. Similarly, $xx_i e_i x_0 e_j x_1 \cdots x_{2k-1} y$ for $i = 1, 2, \ldots, m, j = n + 1, \ldots, 2k - 1$ and $xx_i x_2 \cdots x_m x_0 x_{n+1} \cdots x_{2k-1}$ determines $m + 2k - n$ paths of the same parity and of different lengths. The two path systems have $m + 2k$ paths together therefore one of them have at least $k + 1$ paths unless $m = 0$. But in this case $n = k$ and we can apply the same argument by exchanging the role of $x$ and $y$. □

Lemma 4. Assume that $A$ is adjacent to $p$ vertices, $y_1, y_2, \ldots, y_p$ of $C$ and to $q + 1$ vertices of $S$. Moreover, $B$ is adjacent to $y \in V(C) - \{y_1, \ldots, y_p\}$. Then $|L(G)| \geq \lfloor p/2 \rfloor + q$.

Proof. Assume that $y_1, y_2, \ldots, y_p$ and $y$ follow each other in this order along $C$. Let $Az_1, Az_2, \ldots, Az_q$ be the diagonals of $S$ in this order starting from $A$. We shall define $q + 1$ paths $S_i$ and $p$ paths $P_i$ as follows:

$$S_i = S, \quad S_{i+1} = A, z_i, z_{i+1}, \ldots, z_q, B \quad \text{for } i = 1, 2, \ldots, q$$

$$P_i = A, y_i, y_{i+1}, \ldots, y_p, y, B \quad \text{for } i = 1, 2, \ldots, p.$$ 

Let $a$ denote the length of the subpath $z_qB$ on $S$. It is easy to check that the following numbers are all odd cycle lengths:

$$f_i = |S_i| - a + 1 \quad \text{if } |S_i| - a \text{ is even,}$$

$$g_q = |S_1| + |P_1| \quad \text{if } |S_1| + |P_1| \text{ is odd,}$$

$$h_p = |S_1| + |C| - |P_1| + 2 \quad \text{if } |S_1| + |P_1| \text{ is even.}$$

By the symmetry of (2) and (3) one may assume that $a$ is even.

Let $i, j, m$ be indices such that

$$|S_i| \text{ is even,} \quad |S_j| \text{ is odd,} \quad |P_m| \text{ is even.}$$

If $f_i \geq g_j$, i.e., $|S_i| - a + 1 \geq |S_j| + |P_m|$ then (3) implies that

$$h_{im} = |S_i| + |C| - |P_m| + 2 \geq |S_j| + |P_m| + a - 1 + |C| - |P_m| + 2$$

$$= |S_j| + 1 + a + |C| > |C|$$
contradicting the maximality of \( C \). Therefore \( f_i < g_{jm} \) for each choice of indices satisfying (4). Thus \( g_{jm} \) takes at least \( J + M - 1 \) values where

\[
J = \{|j: |S_j| \text{ is odd}|\}, \quad M = \{|m: |P_m| \text{ is even}|\}.
\]

If \( I = \{|i: |S_i| \text{ is even}|\} \) then \( f_i < g_{jm} \) implies that \( f_i \) and \( g_{jm} \) together take at least \( q + M \) values, i.e.,

\[
|L(G)| \geq q + M.
\]  

Similarly, let \( i, j, n \) be indices such that

\[
|S_i| \text{ is even}, \quad |S_j| \text{ is odd}, \quad |P_n| \text{ is odd}.
\]  

If \( f_i > h_{jn} \), i.e., \( |S_i| - a + 1 \geq |S_j| + |C| - |P_n| + 2 \) then (2) implies

\[
g_{in} = |S_i| + |P_n| \geq |S_j| + |C| - |P_n| + 2 + a - 1 + |P_n| = |S_j| + a + 1 + |C| > |C|
\]

leading to a contradiction. Therefore \( f_i < h_{jn} \) for each choice of indices satisfying (6). Thus for \( N = \{|n: |P_n| \text{ is odd}|\} \) we get

\[
|L(G)| \geq q + N.
\]  

Since \( M + N = p \), either \( M \) or \( N \) is at least \( \lfloor p/2 \rfloor \) and the lemma follows from (5) and (7). \( \Box \)

**Lemma 5.** If \( V(G) - V(C) \) is an independent set then \( G = K_{2k+2} \).

**Proof.** If \( V(G) - V(C) = \emptyset \) then \( |L(G)| \geq k + 1 \) follows from Lemma 2 and we have a contradiction. Select a vertex \( T \in V(G) - V(C) \). Then \( d(T) \geq 2k + 1 \) implies that \( T \) is adjacent to \( X = \{x_1, x_2, \ldots, x_{2k+1}\} \subseteq V(C) \) and assume that the vertices of \( X \) follow each other in this order. This order gives an orientation to \( C \). Set \( A_i^x = a_1^x + a_2^x + \cdots + a_i^x \) where \( x = x_m \) and \( a_i^x \) is the length of the path on \( C \) from \( x_{m+j-1} \) to \( x_{m+j} \), \( i = 1, 2, \ldots, 2k + 1 \). Here indices are taken modulo \( 2k + 1 \) and the path is understood according the fixed orientation of \( C \). Clearly \( A_{2k+1}^x = |C| \) for any \( x \in X \). Moreover, for any \( x \in X \) we have the following odd cycle lengths:

\[
A_i^x + 2 \quad \text{if } A_i^x \text{ is odd and } 1 \leq i < 2k + 1,
\]

\[
|C| - A_i^x + 2 \quad \text{if } A_i^x \text{ is even and } 1 \leq i \leq 2k + 1.
\]  

For any \( x \in X \) there are \( k + 1 \) elements of \( \{A_i^x\}_{i=1}^{2k+1} \) having the same parity. If they are even then (8) implies that \( |L(G)| \geq k + 1 \), a contradiction. Thus we may assume that for each \( x \in X \), \( \{A_i^x\}_{i=1}^{2k+1} \) contains exactly \( k + 1 \) odd numbers and \( A_i^x + 2 = |C| \) for some \( i \). This implies that \( X \) divides \( C \) into paths of length 1 or 2. If \( A_i^x = 2 \) for some \( x \in X \) then there are \( k \) odd numbers in \( \{A_i^x\}_{i=1}^{2k} \) but all of them are larger than 3. Since \( |C| \) is odd, there exists \( y \in X \), \( A_i^y = 1 \) and we have a 3-cycle giving \( k + 1 \) odd cycle lengths altogether. Thus the only possibility is that
$X$ divides $C$ into paths of length 1, i.e., $|C| = 2k + 1$. Now $V(G) - V(C) = \{T\}$ otherwise we get a cycle of length $2k + 3$, thus $G$ has odd cycle lengths $2i + 1$ for $i = 1, 2, \ldots, k + 1$, leading to a contradiction. Thus $|V(G)| = 2k + 2$ and since each degree of $G$ is at least $2k + 1$, $G = K_{2k + 2}$.

**Lemma 6.** If $S$ has one diagonal at $A$ and one at $B$, and, moreover, $\Gamma(A) \cap C = \Gamma(B) \cap C$, $|\Gamma(A) \cap C| = 2k - 1$ then $|L(G)| \geq k + 1$.

**Proof.** Let $\Gamma(A) \cap C = \Gamma(B) \cap C = X = \{x_1, x_2, \ldots, x_{2k-1}\}$ and assume that $x_1, x_2, \ldots, x_{2k-1}$ is their order on $C$. We use the notation $A^i_x$ for $i = 1, 2, \ldots, 2k - 1$ as defined in Lemma 5. First we assume that $k \geq 2$.

Let $W$ denote the graph consisting of $S$ with its two diagonals. Let $R(W)$ denote the set of path lengths in $W$ from $A$ to $B$. It is easy to check that $|R(W)| \geq 3$ except when $R(W) = \{b + 2, 3b + 2\}$ or when $R(W) = \{2, b + 2\}$ for some $b$ (in these cases $S$ has two crossing diagonals and $b$ is the length of the middle segment of $S$).

If $x, y \in X$, $x \neq y$, then $x$ and $y$ can be connected by paths of length $2$ and $h + 2$ for $h \in R(W)$. Therefore we have the following odd cycle lengths:

\[
\begin{align*}
A^i_x + 2, A^i_y + h + 2 & \quad \text{if $A^i_x$ is odd and $h \in R(W)$ is even,} \\
A^i_x + h + 2 & \quad \text{if $A^i_x$ is even and $h \in R(W)$ is odd,} \\
|C| - A^i_x + 2, |C| - A^i_y + h + 2 & \quad \text{if $A^i_x$ is even and $h \in R(W)$ is even,} \\
|C| - A^i_x + h + 2 & \quad \text{if $A^i_x$ is odd and $h \in R(W)$ is odd.}
\end{align*}
\]

The index $i$ in (9) can take values $1, 2, \ldots, 2k - 2$.

If there are at least $k A^i_x$ of the same parity for some $x \in X$ then selecting two numbers from $\{2\} \cup \{h + 2: h \in R(W)\}$ of the same parity, (9) implies $|L(G)| \geq k + 1$, contradiction. Assume that $A^i_x$ is even for some $x \in X$. Then $A^i_y, \ldots, A^{2k-3}_y$ contain $k - 1$ numbers of the same parity where $y$ is the vertex in $X$ following $x$. Adding $A^i_y$ to the largest of these numbers we get $k$ different numbers of the same parity. Therefore $A^i_x$ is odd for each $x \in X$. Consequently

\[
\min\{A^i_x: x \in X\} = \min\{A^i_y: x \in X\}
\]

is an odd number thus $\min\{A^i_x: x \in X\} + 2$ and $A^i_x + h + 2$ gives $k + 1$ odd numbers if $A^i_x$ is even and $h$ is odd, provided that $R(W)$ has two odd numbers. If $|R(W)| \geq 3$ then this is true because $\{2\} \cup R(W)$ must contain two odd and two even numbers (three of the same parity would give $k + 1$ odd cycles by the second or fourth line of (9)). But if $|R(W)| = 2$ then $W$ is described before. If $R(W) = \{b + 2, 3b + 2\}$ then for even $b$, $\{2\} \cup R(W)$ contains three even numbers, for odd $b$, $R(W)$ contains two odd numbers. If $R(W) = \{2, b + 2\}$ then for even $b$, $\{2\} \cup R(W)$ contains three even numbers. For odd $b$ one can find $k + 1$
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odd numbers as follows:

\[
\min\{A_i^x: x \in X\} + 2, \quad \min\{A_i^z: x \in X\} + 4,
\]

\( A_i^z + b + 4 \) for \( A_i^x \) even.

Finally, the case \( k = 1 \) is treated. Now \( |X| = 1 \) and using the 2-connectedness of \( G \), there is a path from \( \nu(C) - X \) to \( S \). One can easily find two odd cycles of different lengths, the missing details are omitted here. \( \square \)

**Lemma 7.** If \( \Gamma(A) \cap C = \Gamma(B) \cap C \) and \( |\Gamma(A) \cap C| = 2k \), \( A \neq B \) then \( |L(G)| \geq k + 1 \).

**Proof.** Assume that \( \Gamma(A) \cap C = X = \{x_1, x_2, \ldots, x_{2k}\} \) and we shall use the notations of the previous lemmas. Let \( s \) denote the length of \( S \). It is clear that for each \( x \in X \) \( A_i^z \), \( A_j^z \), \( \ldots, A_{2k}^z \) must consist of \( k - 1 \) numbers of one parity and \( k \) numbers of the other parity, otherwise \( |L(G)| \geq k + 1 \) follows. It is easy to check that if \( \{A_i^z\}_{i=1}^{2k-1} \) contains \( k - 1 \) odd numbers then \( \{A_i^z\}_{i=1}^{2k-1} \) contains \( k \) odd numbers for \( y = x_{i+1} \) if \( x = x_i \).

If \( s \) is odd then select \( x \) such that \( \{A_i^z\}_{i=1}^{2k-1} \) contains \( k - 1 \) odd numbers, i.e., \( k \) even numbers: \( b_1 < b_2 < \cdots < b_k \). Then we have \( k + 1 \) different odd cycle lengths: \( s + 2 \), \( b_1 + s + 2 \), \( \ldots, b_k + 2 < b_k + 2 + s \) are different odd cycle lengths.

**Lemma 8.** If \( |\Gamma(A)| \cap C = 2k \) and \( y \in (\Gamma(B) \cap C) - \Gamma(A) \) then \( |L(G)| \geq k + 1 \).

**Proof.** Assume that \( \Gamma(A) \cap C = \{x_1, x_2, \ldots, x_{2k}\} = X \) and \( y \equiv x_0, x_1, \ldots, x_{2k} \) follow each other on \( C \) in this order. Let \( a_i \) be the length of the path connecting \( x_i \) and \( x_{i-1} \) on \( C \) which does not contain other \( x_j \). The length of \( S \) is denoted by \( s \).

Set \( A_i = a_1 + \cdots + a_i \) for \( i = 1, 2, \ldots, 2k \). We may assume that \( \{A_i\}_{i=1}^{2k} \) contains \( k \) odd and \( k \) even numbers otherwise \( |L(G)| \geq k + 1 \) is obvious.

**Case 1:** \( a_1 + s = 0 \mod 2 \).

Let \( I \) be the set of those indices for which \( A_i + s + 2 \) is odd. Clearly, \( |I| = k \) and for \( i, j \in I \), \( A_i - a_1 + 2 = A_j + s + 2 \), therefore \( A_i - a_1 + 2 \) and \( A_j + s + 2 \) are odd cycle lengths in \( G \). Let \( A_j \) be the smallest element of \( \{A_i: i \in I\} \), then \( A_j - a_1 + 2 < A_i + s + 2 \) for \( i \in I \) since \( A_j < A_i + s + a_1 \) for \( i \in I \). Therefore \( A_j \) and \( A_i + s + 2 \) for \( i \in I \) gives \( k + 1 \) different odd cycle lengths.

**Case 2:** \( a_1 + s = 1 \mod 2 \).

In this case \( a_1 + s + 2 \) is an odd cycle length. Let \( I \) be the set of indices \( i \) such that \( |C| - A_i + a_1 + 2 \) is odd. Clearly, \( |I| = k \). We claim that \( a_1 + s + 2 \neq |C| - A_i + a_1 + 2 \) for \( i \in I \). If there is equality for some \( i \in I \) then \( A_i = |C| - s \). Since
|C| - A_i + a_1 + 2 is odd, \( A_i \equiv a_1 \mod 2 \) therefore \( A_i + 2 + s = a_1 + 2 + s \equiv 1 \mod 2 \). Thus \( A_i + 2 + s = |C| - s + 2 + s = |C| + 2 \) and \( A_i + 2 + s \) is the length of an odd cycle. This contradicts the maximality of \( C \). Thus \( a_1 + s + 2 \) is different from \( |C| - A_i + a_1 + 2 \) for \( i \in I \) and we have \( k + 1 \) different odd cycle lengths.

Reference