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Three-regular Path Pairable Graphs

Ralph J. Faudree[1]*, András Gyárfás[2], and Jenö Lehel[2]

[1] Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA
[2] Computer and Automation Institute Hungarian Academy of Sciences, Budapest P.O.B. 63, H-1518, Hungary

Abstract. A graph G with at least $2k$ vertices is k-path pairable if for any k pairs of distinct vertices of G there are k edge disjoint paths between the pairs. It will be shown for any positive integer k that there is a k-path pairable graph of maximum degree three.

1. Introduction

We shall consider graphs without loops or multiple edges. Any such graph can quite naturally represent a computer or communication network. There are various reasonable ways to measure the capability of the network represented by this graph to transfer information and handle communications. We will consider the capability of the network to allow messages to be passed simultaneously between any fixed number of pairs of nodes of the network. With this in mind, we give the following formal definition.

Definition. Given a fixed positive integer k, a graph G is *k-path pairable* if for any pair of disjoint ordered sets of vertices $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ of G there are k edge-disjoint paths P_i, where P_i is a path from x_i to y_i, for $1 \leq i \leq k$.

The concept of k-path pairable is related to several other concepts. It is closely related to but is not the same as weakly k-linkable. The definition of weakly k-linkable is the following:

Definition. Given a fixed positive integer k, a graph G is *weakly k-linked* if for any collection of k pairs of vertices (not necessarily distinct pairs) $\{(x_i, y_i): 1 \leq i \leq k\}$ of G, there are k edge-disjoint paths P_i, where P_i is a path from x_i to y_i, for $1 \leq i \leq k$.

In both weakly linked and path pairable graphs k edge disjoint paths are required, but duplication of the pairs is allowed in the weakly linked case and prohibited in the path pairable case. By definition, any k-linked graph is k-pairable. However, any graph that is weakly k-linked is clearly at least k-edge connected. It

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has been shown by Andreas Huck [7] that any $(k+2)$-edge-connected graph is k-linked. This edge connectivity condition is not implied by k-path pairable. For example, it can be shown that the Petersen graph on 10 vertices is 5-path pairable, but the graph is regular of degree three. In fact, it will be shown for any positive integer k that there is a k-path pairable graph of maximum degree three.

There are vertex versions of the previous concepts which are stronger. Given a positive integer k a graph G is *k-linked* if given any collection of k pairs of vertices there are k vertex disjoint paths, one between each of the k pairs of vertices. It was proved independently by [8] and [6] that any $3k \cdot 2^{\binom{3k}{2}}$ vertex connected graph is k-linked. Another related concept is that of *superconcentrator*, which is a bipartite graph with the property that between any two sets of the same cardinality d in different parts of the bipartite graph, there are d vertex disjoint paths between the corresponding sets of vertices. It has been shown that there are sparse graphs that are superconcentrators (see [4], [10], and [11]).

Let $p_k(n, \Delta)$ be the minimum number of edges in a graph G of order n and maximum degree at most Δ that is k-path pairable. Our objective is to evaluate the function $p_k(n, \Delta)$. Useful in the determination of this function is the function $p_k(n, \Delta, \delta)$, which is the minimum number of edges in a graph G of order n with maximum degree at most Δ and minimum degree at least δ that is k-path pairable. We start with some trivial observations.

Any connected graph is 1-path pairable, so $p_1(n, \Delta) = n - 1$ for any $2 \leq \Delta < n$. The star $K_{1,n-1}$ is k-path pairable for any $k \leq n/2$, so $p_k(n, n-1) = n - 1$ for all $k \leq n/2$. More generally consider the graph obtained from a K_m by attaching stars with $(n-m)/m$ edges on each vertex of the K_m (assume m divides n). This graph has n vertices, $n + \binom{m-1}{2} - 1$ edges, maximum degree $\frac{n}{m} + m - 2$, and it is easy to verify that it is $(m-1)$-path pairable. Thus for k and l fixed with $l \geq k+1$ and n sufficiently large, $p_k(n, n/l) \leq n + c$ for some $c = c(k, l)$. However, when there are additional restrictions placed on the maximal degree of the graph (the maximum degree is not a positive fraction of the number of vertices), then the number of edges required will be more.

In [5], we deal with the cases $k = 2$ and 3. We prove, among others, that for $\Delta \geq 9$ a fixed integer and n sufficiently large, $p_k(n, \Delta) = (1 + \varepsilon)n$, where ε depends upon Δ and approaches 0 as Δ increases. For $k > 3$ the nature of minimal k-path pairable graphs differs from those for $k = 2$ or 3. Here we prove the following theorem which illustrates this

Theorem. *Let $k > 3$ be a fixed integer. For n sufficiently large there exist constants ε_1 and ε_2 (that depend upon k and approach* 0 *as k increases) such that*

$$(\tfrac{3}{2} - \varepsilon_1)n < p_k(n, 3) < (\tfrac{3}{2} - \varepsilon_2)n.$$

It would be nice to be able to get a sharp bound for $p_k(n, 3)$ for $k > 3$ similar to what was done for $k = 2$ or 3 in [5]. Also, it would be interesting to show for any fixed positive integer $\Delta > 3$ that $p_k(n, \Delta)$ is essentially the same as $p_k(n, 3)$, and the maximum degree hypothesis of the Theorem could be changed from 3 to Δ.

We will generally follow the notation of [3]. For a graph G, the vertex and edge set will be denoted by $V(G)$ and $E(G)$ respectively. The cardinality of $V(G)$ and $E(G)$ will be called the *order* and *size* respectively of the graph G. If X is a collection of vertices and edges of G, then $G - X$ will denote the graph obtained from G by deleting the edges in X and by deleting the vertices in X and the edges incident to a vertex in X. If u and v are vertices in G, then the edge determined by this pair of vertices will be denoted by uv.

2. Upper Bound

For any fixed positive integer k we describe a graph of large order n that has maximum degree three and is k-path pairable. We start with some preliminary observations.

Let B_r denote the complete binary tree with r levels in which the root has degree three as well. Therefore, B_r has $3 \cdot 2^r - 2$ vertices and $3 \cdot 2^{r-1}$ leaves. For each nonleaf v of the tree B_r consider the unique path P_v from v to a leaf of the tree by first taking the right tree rooted at v and the left tree at each subsequent step. It is straightforward to verify that distinct vertices of the tree determine edge disjoint paths to appropriate leaves of the tree. Therefore, for any collection of t nonleaf vertices of the tree, there are t edge disjoint paths from these vertices to t distinct leaves of the tree. Also, note that if the t vertices are all in the first s levels of the tree ($s < r$), then the endvertices of the t paths associated with these vertices will be pairwise at a distance of at least $2(r - s)$, since the last $r - s$ vertices in each of these paths will be in different rooted binary trees with $r - s$ levels.

Also note that the same conclusion is true for the complete binary tree with a root of degree two instead of a root of degree three. Furthermore, a vertex of degree one can be attached to the root of degree two of this tree, and a path that always takes the left trees can be associated with this new vertex. This gives 2^r edge disjoint paths from the nonleaves and the new vertex to the 2^r leaves of the original binary tree.

If a graph G has sufficiently high vertex connectivity, then G will be k-linked. More specifically, recall that a connectivity of $3k \cdot 2^{\binom{3k}{2}}$ is sufficient for a graph to be k-linked. With that in mind, we will describe a graph that will have sufficient connectivity to have the linked properties needed in the proof of the next theorem.

For fixed positive integers $d \leq g$, and for n sufficiently large there exists a graph of order n that is regular of degree d, is d-connected, and has girth at least g (see [1]). Let $G_n(d, g)$ be a graph with this property. If d is even, then the graph has a 2-factorization by Petersen's Theorem [9]. Thus the edges of $G_n(d, g)$ can be colored with $d/2$ colors such that each monochromatic subgraph is a disjoint union of cycles, and each vertex of $G_n(d, g)$ will be incident to precisely two edges of each color.

Consider the case when $d = 3 \cdot 2^r$, so the edges of $G_n(3 \cdot 2^r, g)$ are colored with $3 \cdot 2^{r-1}$ colors. Associate with each vertex of $G_n(3 \cdot 2^r, g)$ a distinct binary tree B_r. The $3 \cdot 2^{r-1}$ leaves of each of these complete binary trees B_r can be colored with the same $3 \cdot 2^{r-1}$ colors. Now, a new graph $G_n^*(3 \cdot 2^r, g)$ that is regular of degree three can be constructed from $G_n(3 \cdot 2^r, g)$ by replacing each vertex v with $3 \cdot 2^{r-1}$ vertices, one associated with each of the $3 \cdot 2^{r-1}$ colors, and making each of these vertices incident to the two edges of that color that were incident to v. Now identify each colored leaf of the binary tree associated with v with the new vertex derived from v associated with the same color. The graph $G_n^*(3 \cdot 2^r, g)$ has $(3 \cdot 2^r - 2)n$ vertices, all of which have degree three.

This graph has two kinds of edges: there are the *tree edges* that came from the attached binary trees, and there are the *graph edges* that are identified with edges in the initial graph $G_n(3 \cdot 2^r, g)$. The girth of the graph $G_n^*(3 \cdot 2^r, g)$ is at least g, since any cycle in this graph determines a closed Eulerian circuit in $G_n(3 \cdot 2^r, g)$ and each of the cycles associated with that circuit has at least g vertices. The graph $G_n(3 \cdot 2^r, g)$ will be called the *core* graph of the graph $G_n^*(3 \cdot 2^r, g)$, and the binary trees B_r associated with each vertex of the core will be called the *attached* trees. These graphs are central to the proof of the following theorem.

Theorem 1. *For any positive integer k and for n sufficiently large there is a graph of order n and regular of degree three that is k-path pairable.*

Proof. For $k = 1$, any connected 3-regular graph will suffice, so we will assume that $k \geq 2$. Select any integer r such that $r \geq 9k^2/2$. Consider a graph $G^* = G_n^*(3 \cdot 2^r, 3 \cdot 2^r)$, and let G denote the core graph of G^*. Let $X = \{x_1, x_2, \ldots, x_{2k}\}$ be a set of $2k$ vertices in G^* with x_{2i-1} paired with x_{2i} for $1 \leq i \leq k$. We will show that there are edge disjoint paths between these k pairs of vertices.

We will first partition the vertices in X such that the vertices in the same partition class are close and vertices in different partitions are far apart. More specifically, we will find an integer s with $2r \leq s \leq (2r)6^{2k-1} = 6^{2k}r/3$ and a subset T of X such that each vertex of X is within a distance s of precisely one of the vertices of T and the distance between vertices in T is at least $5s$.

To verify the existence of this partition, first try $s_1 = 2r$, and $T_1 = X$. If each pair of vertices in X are a distance at least $5(2r)$ apart, we have the desired partition. If not, then let $s_2 = 6s_1 = 12r$. If two vertices of T_1 are within a distance $5(2r)$ of each other, then one of these vertices will not be needed to cover the set X if we use a distance $6(2r)$. Thus, we can select a proper subset T_2 of T_1 such that each vertex of X is within a distance s_2 of the vertices of T_2. If this gives the desired partition, we are done. If not, then there are two vertices in T_2 that are within a distance $5s_2$ of each other. Therefore, if we let $s_3 = 6s_2$, then there is a proper subset T_3 of T_2 such that each vertex of X is within a distance s_3 of a vertex in T_3. Since there are only $2k$ vertices in X, within $2k$ steps this procedure will yield the appropriate subset T and integer s.

Let $\{y_1, y_2, \ldots, y_t\}$ be the vertices of T, and let X_i be the subset of X of those vertices that are within a distance s of y_i, for $1 \leq i \leq t$. If, for any i, we consider the vertices of G that are within a distance $2s$ of y_i, then we have a binary tree (with a root of degree three) since the graph is 3-regular and the girth g is large ($g > 4s$). Therefore from the previous observations about binary trees, there are for each i, edge disjoint paths of lengths between s and $2s$ from each of the vertices in X_i, such that the terminal vertices of these paths are pairwise at a distance of at least $2s$. Because the distance between vertices in T is at least $5s$, there are edge disjoint paths from all of the vertices in X such that the terminal vertices of these paths are pairwise at a distance at least s. Let $\{P_1, P_2, \ldots, P_{2k}\}$ denote these paths of length at most $2s$

and let $\{w_1, w_2, \ldots, w_{2k}\}$ be the collection of these endvertices of the paths from $\{x_1, x_2, \ldots, x_{2k}\}$ respectively.

Because of the distance between the vertices in $\{w_1, w_2, \ldots, w_{2k}\}$, each of the vertices w_i is in a different attached binary tree of G^* that was associated with the vertices of the initial core graph G. We can also assume that each w_i is a leaf of this attached binary tree. For each i, let w_i' be the vertex in the core graph G that is associated with the attached tree containing w_i. In the $2k$ paths from the x_i to the w_i there are a total of at most $4ks + 2k$ vertices. Delete all of the vertices in these paths, except for the terminal vertices $\{w_1, w_2, \ldots, w_{2k}\}$, and in fact delete all of the vertices of any of the attached binary trees that contain vertices in these paths. Let Z^* denote the vertices deleted from G^*. Related to the deletion of the set of Z^* vertices of G^*, there is the deletion of a corresponding set Z of at most $4ks$ vertices in the core graph G.

The graph $G - Z$ is still $3k \cdot 2^{\binom{3k}{2}}$-connected, since r was chosen to satisfy $3 \cdot 2^r - (4k)(6^{2k}r/3) > 3k \cdot 2^{\binom{3k}{2}}$, as the reader can check easily. Therefore by [8] and [6], the graph $G - Z$ is k-linked. Thus in $G - Z$ there are k vertex disjoint paths between the pairs w_{2i-1}' and w_{2i}' for $1 \le i \le k$. This translates into k edge disjoint paths Q_i in the graph $G^* - Z^*$ between the pairs w_{2i-1} and w_{2i} for $1 \le i \le k$. These k paths are edge disjoint from the $2k$ paths P_i for $1 \le i \le 2k$. Combining the paths P_{2i-1}, P_{2i} and Q_i give the desired paths between the pairs x_{2i-1} and x_{2i} for $1 \le i \le k$. This completes the proof of Theorem 1. □

In the proof of Theorem 1, the roots of the attached binary trees were of degree three. However, it is sufficient to attach complete binary trees with roots that are of degree two. In addition, a vertex of degree one can be joined to the root of each of the attached binary trees and the same construction for a path pairable graph will be valid. Only minor obvious adjustments would have to be made in the verification of the path pairable property. If we use the notation of Theorem 1, then in the case when a binary tree with a vertex of degree one joined to the root of the tree is the attached tree, the graph G^* would have $2^{r+1}n$ vertices and all these vertices would have degree three except for n vertices of degree one. Thus G^* would have $(3 \cdot 2^r - 1)n$ edges. Thus as a consequence of the proof of Theorem 1 we have the following corollary.

For $k > 3$ fixed and n sufficiently large, there is a graph of maximum degree three with n vertices and $\frac{3}{2}\left(1 - \frac{1}{3 \cdot 2^r}\right)n$ edges (where $r = 9k^2/2$) that is k-path pairable.

Theorem 2. *Let $k > 3$ be a fixed integer. For n sufficiently large,*

$$p_k(n, 3) < \frac{3n}{2}\left(1 - \frac{1}{3 \cdot 2^{9k^2/2}}\right).$$

3. Necessary Conditions

We begin with some general observations about graphs that are k-path pairable. Let G be a k-path pairable graph, and let X be a set of cut edges of G that separates the vertices into two sets C_1 and C_2. If t vertices in one part are paired with t vertices in the other part, then there must be at least t edges in X. Thus,

$$|X| \geq \min\{|C_1|, |C_2|, k\}.$$

This condition is called the *Cut Condition* for a k-path pairable graph, and is clearly a necessary condition for a graph to be k-path pairable. However, as we shall see later, it is not sufficient.

There are several immediate consequences of the Cut Condition. Let H be a connected subgraph of order m for $m \leq k$ of a k-path pairable graph G that has maximal degree Δ, and assume that H contains m_i vertices of degree i in G for $i = 1, 2$. Then, the number of edges emanating from H is at most $m\Delta - (\Delta - 2)m_2 - (\Delta - 1)m_1 - 2m + 2$ (the extreme case is when H is a tree and all of the vertices are of degree 1, 2, or Δ), and this number must be at least m by the Cut Condition. Therefore,

$$(\Delta - 2)m_2 + (\Delta - 1)m_1 \leq (\Delta - 3)m + 2.$$

In particular, this implies that a vertex of degree d cannot be adjacent to $\left\lceil \frac{d}{2} \right\rceil$ vertices of degree 1 for $d \leq 2k - 2$. Also, if the maximum degree of G is 3, then in any collection of k vertices that form a connected subgraph there will be at most one vertex of degree one (and no other vertices of degree less than 3) or at most two vertices of degree two (and no vertices of degree one). This implies that in such a graph G no vertex of degree less than 3 can be within a distance $k - 1$ of a vertex of degree one, and at most one vertex of degree less than 3 can be within a distance $(k - 1)/2$ of a vertex of degree two. A *suspended path* in a graph is a path in which all of the interior vertices have degree two in the graph. For $k \geq 3$, the graph G cannot have a suspended path with 5 vertices (three interior vertices of degree two), because this would imply the existence of two vertices within a distance one of a vertex of degree two.

The Cut Condition implies that certain induced subgraphs are forbidden in a k-path pairable graph. There are, however, other forbidden subgraphs that are not implied by the Cut Condition. For example, for every $k > 1$ the graph of the $2k$-dimensional cube is not k-path pairable, however it can be shown that satisfies the Cut Condition (see [2]).

4. Lower Bound

Theorem 2 together with Theorem 3 proved in this section conclude the proof of the main result stated in the first section.

Theorem 3. *Let $k > 3$ be a fixed integer. For n sufficiently large,*

$$\frac{3n}{2}\left(1 - \frac{4}{3(k + 1)}\right) < p_k(n, 3).$$

Proof. Consider a graph G of order n that is k-path pairable. Let n_1, n_2, and n_3 be the number of vertices of degree 1, 2, and at least 3 respectively in the graph G. To each vertex v_1 of G of degree one identify the vertices in G that are within a distance $\lceil (k-1)/2 \rceil$ of v_1, and to each vertex v_2 of degree two identify the vertices of G than are within a distance $\lceil (k-1)/3 \rceil$ of v_2. Any vertex of G identified with a vertex of degree one is not identified with any other vertex.

Also, no vertex of G can be identified with more than two vertices, for if a vertex were identified with three vertices then there would be either a vertex of degree less than three within a distance $k-1$ of a vertex of degree one, or there would be three vertices of degree two in a connected graph with at most k vertices. These are forbidden structures in a k-path pairable graph. With each vertex of degree one there are at least $(k-1)/2$ vertices of degree at least three identified with this vertex, and with each vertex of degree two there are at least $(k-1)/3$ vertices of degree at least three identified with this vertex. Hence we have

$$n_3 \geq \frac{(k-1)}{2} n_1 + \frac{(k-1)}{6} n_2$$

vertices of degree three. Also, the average degree in G is at least $(n_1 + 2n_2 + 3n_3)/2(n_1 + n_2 + n_3)$. Thus, the average degree in G is at least

$$\frac{n_1 + n_2 + 3(k-1)n_1/2 + 3(k-1)n_2/6}{2(n_1 + n_2 + (k-1)n_1/2 + (k-1)n_2/6}.$$

This reduces to

$$\frac{(3k-1)n_1/2 + (k+1)n_2/2}{2(k+1)n_1/2 + (k+1)n_2/6)} \geq \frac{3}{2}\left(1 - \frac{4}{3(k+1)}\right).$$

This verifies the left hand inequality of (3) and completes the proof of Theorem 3. □

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