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NOTE

EFFECTIVE ON-LINE COLORING OF P_5-FREE GRAPHS

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An on-line algorithm is given that colors any P_5-free graph with $f(\omega)$ colors, where f is a function of the clique number ω of the graph.

A *proper coloring* of a graph is an assignment of positive integers called colors to its vertices such that adjacent vertices have distinct colors.

An *on-line coloring* is an algorithm that colors vertices of a (finite) graph in the following way:

— vertices are taken in some order $v_1, v_2, \ldots$;
— a color c_i is assigned to v_i by only looking at the subgraph G_i induced by $v_1, \ldots, v_i$, $i = 1, 2, \ldots$;
— the color of v_i never changes during the algorithm, $i = 1, 2, \ldots$;
— the obtained coloring is a proper coloring of G_i, $i = 1, 2, \ldots$.

The most common on-line coloring is the *first fit coloring*, **FF**, that at each step assigns the smallest possible integer as color to the current vertex of the graph. The concept of on-line and first fit chromatic number was introduced and investigated recently in [2], [3], [4] and [5]. Here we investigate the problem of effectivity of on-line coloring for a particular family of graphs.

An on-line coloring **A** is said to be *effective* on a family $\mathcal{K}$ if there exists a function $f(\chi)$ such that the number of colors used by **A** for any ordering of $V(G)$ is at most $f(\chi(G))$ for every $G \in \mathcal{K}$, where $\chi(G)$ denotes the chromatic number of G. In most cases a stronger statement is proved, namely that the number of colors is at most $f(\omega(G))$, where $\omega(G)$ is the order of the maximum clique of G.

A graph is called P_k-free if it contains no path on k vertices as induced subgraph. In [2], it is proved that **FF** is perfect for P_4-free graph, i.e., if G is a P_4-free graph, then **FF** colors G by exactly $\chi(G)$ colors.

On the other hand on-line colorings are ineffective on the family of P_6-free graphs: there is a sequence $G_1, G_2, \ldots$ of bipartite P_6-free graphs such that every on-line coloring colors G_n by at least n colors for $n = 1, 2, \ldots$ (cf. [2]).

In this paper we fill the gap by proving the following theorem.

Theorem. *There is an effective on-line algorithm for the family of P_5-free graphs.*

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The vertex set of a graph $G = (V, E)$ is considered as an ordered set, and we assume that vertices are taken by the on-line coloring procedure according to that ordering. We denote by $G[A]$ the ordered subgraph induced by $A \subseteq V$. For $x \in A$ let $A_x = \{v \in A : v \leq x\}$ (in particular $V_x = \{v \in V : v \leq x\}$). Let $G_x = G[V_x]$ and C_x be the component of G_x containing x. Let $\omega(G)$ denote the order of a maximum clique of G.

Our on-line coloring algorithm is a function $c(x, G_x)$ defined by recursion on subgraphs of G_x with smaller clique size. The value of $c(x, G_x)$ is a list of non-negative integers, all less than or equal to $\omega(G)$.

The algorithm maintains certain rooted forests, called "frames" of height at most three on subsets of V. If F is a frame and $x > y$ for every $y \in V(F)$ then we shall define a new frame F_x on $V(F) \cup \{x\}$. For $u \in V(F)$ let $CHAIN(u, F)$ be the list of colors on the unique path from a root of F to u, excluding u itself. (If u is a root of F then $CHAIN(u, F) = 0$.)

The function $c(x, G_x)$ is defined for each $x \in V$ by

$$c(x, G_x) = \begin{cases} (0,0,0) & \text{if } C_x = \{x\}, \\ (\omega(C_x), CHAIN(x, F_x), c(x, G[S_x])) & \text{otherwise,} \end{cases}$$

where $S_x = \{v \in V(C_x) : \omega(C_v) = \omega(C_x) \text{ and } CHAIN(v, F_v) = CHAIN(x, F_x)\}$.

We shall show that $\omega(G_x) > \omega(G[S_x])$ which implies that the coloring procedure terminates in at most $\omega(G_x)$ steps. Clearly, a proper coloring of G is obtained.

Let G be a P_5-free graph. Then $G_x - x$ does not have two components each of which contains both neighbors and non-neighbors of x. If $G_x - x$ has one such component, call it $MCOM(x, G_x)$ (main component), otherwise $MCOM(x, G_x) = \emptyset$. Let $LCOM(x, G_x)$ (lessor component) be the union of all components of $G_x - x$ which only contain neighbors of x. The key idea used in the construction of frames is the obvious fact that $\omega(C_x) > \omega(LCOM(x, G_x))$.

We say that F is a *frame* of G if it is a spanning subforest of G satisfying that

(i) each component of F is a rooted tree;
(ii) paths of F starting at roots are induced paths of G;
(iii) if x and y are vertices from different components of F, then xy is not an edge of G.

If F is a frame with vertex set A then F is also called a *frame on A*. The empty set is also considered a frame.

For colored frames an important property (Brothers' rule) is maintained. Let $x \in A$ and F be a frame on $A_x - \{x\}$. Assume for color $c(u) = c(u, G[A_u])$ is defined for every $u \in A_x$.

Brothers' rule. *If F is a colored frame and u and v are inner vertices and brothers in F then $c(u) \neq c(v)$.*

The basic step for the construction of frames is to define the frame F_x on A_x. We use the following notation.

$FATHER(x, F)$ denotes a vertex y of $M = MCOM(x, G[A_x])$ adjacent to x and satisfying:

(a) The path from the root r of M to y contains no vertex but y that is adjacent to x.
(b) if there are inner vertices with property (a), then y is an inner vertex at minimum distance from r in M.

(c) if (b) does not hold, then y is a leaf at a maximum distance from r among leaves of M with property (a).

If $MCOM(x, G[A_x]) = \emptyset$ then set $FATHER(x, F) = \emptyset$. Note that frame property (ii) remains true when xy is added to F as a pendant edge.

Then frame F on $A_x - \{x\}$ is extended to a frame F_x on A_x as follows.

1. The sons of x in F_x are the vertices of $LCOM(x, G[X_x])$ and x becomes the son of $FATHER(x, F)$. If $FATHER(x, F) = \emptyset$ then x becomes a new root of F_x.
2. In order to maintain Brothers' rule the current F_x being is modified according to steps 3 or 4 or both.
3. ($FATHER(x, F) = y$ becomes inner.) If Brothers' rule is violated because y becomes an inner vertex in F_x, then there is a brother y' of y in F such that $y \neq y'$, y' is inner and $c(y) = c(y')$. Let z be a son of y' and let t be the father of y and y'. Since y is defined according to (c), it follows that $xt \notin E(G)$, $xy' \notin E(G)$, $xz \notin E(G)$. Also $yy' \notin E(G)$ because $c(y) = c(y')$ and $zt \notin E(G)$ by definition of a frame.
   Since G is P_5-free, $zy \in E(G)$ follows. Thus all sons of y' in F are adjacent to y in G. Therefore we can modify F_x by replacing the edges $y'z$ with yz for all sons z of y'. Notice that in this step the sons of y' gain a new father (namely y) having the same color as the old one.
4. (x becomes inner.) If Brothers' rule is violated because x become an inner vertex in F_x ($LCOM(x, G[A_x]) \neq \emptyset$) then there exists an inner vertex x' in F such that x' is a brother of x in F_x and $c(x') = c(x)$. Since the sons of x in F_x are not adjacent to any vertex of the path $zx'y$, by definition of a frame, $zx \in E(G)$ follows as in step 3. So it is possible to modify F_x by replacing the edges $x'z$ with xz for all sons z of x'. This modification again preserves the color of the father of z (as in step 3).

Let C_x be the component of $G[A_x]$ containing x. We show that $\omega(G[A_x]) > \omega(G[S_x])$, where S_x is the set of all vertices of C_x having a color with first and second fields identical to that of $c(x)$.

By the definition of F_x, the first field of $c(x)$, i.e., $\omega(C_x)$, is larger than the first field of the color of any vertex in $LCOM(x, G[A_x])$. Therefore $S_x - \{x\} \subset MCOM(x, G[A_x])$. Assume that $u \in S_x$, $u \neq x$. The second fields of $c(x)$ and $c(u)$ are equal, i.e., $CHAIN(x, F_x) = CHAIN(u, F_x)$. Then Brothers' rule implies that x and u are both brothers in F_x. Thus S_x is a subset of the sons of $FATHER(x, F_x)$ and $\omega(G[A_x]) > \omega(G[S_x])$ follows.

To prove the main theorem we show that the number of colors used in our coloring is bounded by a function of $\omega = \omega(G)$. Assume that any graph H with $\omega(H) < \omega$ is colored by at most $f(\omega - 1)$ colors.

Let F be the final frame on $V(G)$. It is enough to show that a component of F has at most $f(\omega)$ colors, since on distinct components the same set of colors is used. One may assume that F has only one component.

Let x be an inner vertex of F and let L be the set of all sons of x in F. Then $L = A \cup B$, where $A = \{y \in L \ : \ y < x\}$ and $B = \{y \in L \ : \ y > x\}$. Since $\omega(G[A]) < \omega$ and $A \subset (V_x - \{x\})$, A is colored with at most $f(\omega - 1)$ colors. If $y \in B$, then the first field of $c(y)$ is at least 2 and at most ω; the second field is the same for all $y \in B$, and the third field can have at most $f(\omega - 1)$ values. Thus L is colored with at most $\omega \cdot f(\omega - 1)$ colors.

Using the fact that F is a tree of height at most 3 follows that F (together with its root) is colored with at most $f(\omega) \leq \omega^3 \cdot f^3(\omega - 1) + 1$ colors. ∎

Notice that we could not decide whether there are simpler non-recursive algorithms, perhaps **FF** is effective to color a P_5-free graph.†

Sumner proved in [6] that if G is P_5-free and $\omega(G) = 2$, then $\chi(G) \leq 3$, and in [1] it is shown that $\chi(G) \leq 4^{\omega-1}$, where $\omega = \omega(G)$. Sumner's result has the following sharper form.

Proposition. *If G is a P_5-free graph with no triangle, then* **FF** *colors G with at most 3 colors.*

Proof. Let G be a P_5-free graph, $\omega(G) = 2$. Assume that **FF** colors G with $k \geq 4$ colors. Let A_i be the set of vertices colored with i, $1 \leq i \leq 4$. Since **FF** is perfect on P_4-free graphs (see [2]), there is an induced path (x_1, x_2, x_3, x_4) in the subgraph of G induced by $V(G) - A_1$. By definition of **FF**, $B_i = \Gamma(x_i) \cap A_i \neq \emptyset$ for each i, $2 \leq i \leq 4$ ($\Gamma(x)$ denotes the set of all vertices adjacent to x in G). Since G is P_5-free, we can find $x \in B_1 \cap B_3$ and $y \in B_2 \cap B_4$. Now $x \neq y$ follows from $\omega(G) = 2$, thus (y, x_4, x_3, x, x_1) is an induced P_5 in G, a contradiction. ∎

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† Note added in proof: H. A. Kierstead, S. G. Penrice and W. T. Troffer answered this affirmatively.