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First fit and on-line chromatic number of families of graphs

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ABSTRACT. On-line proper coloring can be viewed as a two-person game of GraphDrawer and GraphPainter. First both agree on some kit K, a finite or infinite family of graphs they are going to play on. Drawer succesively reveals vertices of a graph with all the edges to earlier vertices, and in each step Painter colors the current vertex.

The aim of Painter consists in using as few distinct colors as possible. Define $\chi^*(K)$ to be the minimum number of colors Painter must use when playing on K. Then $\chi^*(K)$ is called the on-line chromatic number of the family K.

The problem of characterizing graphs with on-line chromatic number not exceeding a fixed integer is investigated. We completely characterize trees with on-line chromatic number at most k for every k.

The best known on-line coloring is the first fit coloring that assigns at each step the smallest possible integer as color to the current vertex of the graph. We also investigate the effectiveness of the first fit coloring on the family of F-free graphs where F is a forest.

1. Introduction

A **proper coloring** of a graph is an assignment of positive integers called colors to its vertices such that adjacent vertices have distinct colors.

An **on-line coloring** is an algorithm that colors vertices of a (finite) graph in the following way:

- vertices are taken in some order $v_1, v_2, \ldots$;

- A color c_i is assigned to v_i by only looking at the subgraph G_i induced by $\{v_1, \ldots, v_i\}, i = 1, 2, \ldots$;

- the color of v_i never changes during the algorithm, $i = 1, 2, \ldots$;

- the obtained coloring is a proper coloring of $G_i, i = 1, 2, \ldots$.

On-line coloring can be viewed as a two-person game of **GraphDrawer** and **GraphPainter**. First of all both agree on some **kit**, a finite or infinite family of graphs they are going to play on. Drawer's moves consist in successively revealing vertices of a graph from the kit with all the edges to earlier vertices, and in each step Painter colors the current vertex thus personalizing an on-line coloring. Tha aim of Painter might be using as few distinct colors as possible and the strategy of Drawer against Painter consists in finding the most challenging piece of the kit that is a worst possible order of vertices of a graph from the family that forces as much colors as possible.

Note that Drawer has the opportunity of 'cheating' (in fact, he is expected to do so): he needs not to fix any particular graph of the family, it's enough to mentain Painter's belief in a consistent game. This means that for every graph G_i induced by $\{v_1, \ldots, v_i\}$ (i.e., the graph Painter can see in the ith step) there exists $H_i \in K$ containing G_i as an induced subgraph, $i = 1, 2, \ldots$.

Let K be a family of graphs. Define $\chi^*(K)$ to be the minimum number of colors Painter must use when playing on K. Then $\chi^*(K)$ is called the **on-line chromatic number** of the family K.

For example, a result by Kierstead and Trotter in [4] can be stated in terms of on-line chromatic number as follows: $\chi^*(K) = 3k - 2$ for the family K of all interval graphs with chromatic number at most k.

Let A be a given on-line coloring. If Painter is restricted to use A when playing on K, then the minimum number of colors Painter must use is called the **A chromatic number** of the family K and is denoted by $\chi_A(K)$.

Note that when restricted to some fixed kit K on-line colorings are **deterministic**: $\chi_A(G') = \chi_A(G)$ if $G' \approx G$, and $G', G \in K$, furthermore **monotone**: $\chi_A(H) \leq \chi_A(G)$ if H is an induced subgraph of G, $G \in K$.

The best known on-line coloring is the **first fit coloring**, FF, that assigns at each step the smallest possible integer as color to the current vertex of the graph.

In the case when the kit consists of a single graph G, we will say that $\chi_A(G) = \chi_A(\{G\})$ is the A chromatic number and $\chi^*(G) = \chi^*(\{G\})$ is the on-line chromatic number of G. Note that $\chi_A(G)$ is the maximum number of colors produced by A for all orderings of the vertices of G, and

$$\chi^*(G) = min\{\chi_A(G) : A \quad \text{is an on-line coloring}\}.$$

A given on-line coloring A is said to be **effective** on a family K if there exists a function $f(\chi)$ such that $\chi_A(G) \leq f(\chi(G))$ for every $G \in K$, where $\chi(G)$ denotes the chromatic number of G. On-line colorings are effective on K if $\chi^*(G) \leq f(\chi(G))$ for every $G \in K$.

In this paper we present some results describing the limit of the power of FF among on-line colorings. In section **3** we characterize trees with on-line chromatic number k for every $k = 1, 2, \ldots$ (Theorem 5). In particular, we obtain that $\chi^*(T) = \chi_{FF}(T)$ for every tree T wich means that FF is as good as any other on-line algorithm on the family of trees.

We show that FF is not effective on the family of permutation graphs (Theorem 2). A result by Kierstead in [3] imply that on-line colorings are effective on permutation graphs if a transitive orientation of the component is given. It would be interesting to get rid of this orientation. It was known that FF is effective (in fact it is perfect) on P_4 - free graphs but on-line algorithms are not effective on P_6 - free graphs (see [1]). We almost decide here the effectiveness of FF on the family of F - free graphs where F is a forest: the only unsettled case is $F = P_5$ (Proposition 8 and Theorem 9). However, we can prove (see [2]) that on-line colorings are effective on the family of all P_5 - free graphs.

2. On-line chromatic number of families

Our first observation points out the difference between the two possible ways of considering on-line colorings as two-person game. There is a striking difference whether Painter knows the graphs which they are playing on or, as in our definition, he only knows the kit, the graph belongs to.

In the following result we show a family K such that $max\{\chi^*(G) : G \in K\} \neq \chi^*(K)$.

Theorem 1. *When playing on the family of all graphs with on-line chromatic number three Drawer has a strategy forcing Painter to use four colors.*

Proof. Suppose that $v_1, v_2, \ldots$ is the ordering of the vertices given by Drawer's successive moves, let G_i be the subgraph induced by $\{v_1, \ldots, v_i\}$ and denote by c_i the color of v_i assigned by Painter.

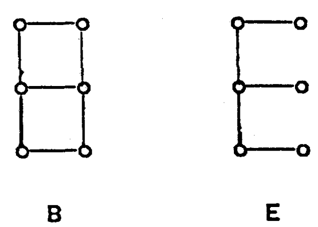

Figure 1.

Let B and E be graphs in Fig. 1. Obviously, $\chi_{FF}(E) = 3$ and it is easy to check that $\chi_A(B) = 3$ for the on-line coloring A as follows:

use FF to color v_i for $i = 1, 2$ and 3;

if $G_4 \approx 2P_2$, then let $c_4 = 3$ and use FF for $i = 5$ and 6,

otherwise use FF for $i = 4, 5$ and 6.

Drawer's 'winning' strategy that forces the using of 4 colors is based on the fact that Drawer is allowed not to fix any particular graph in advance. In fact, he may delay his choice between B and E both contained in the kit until the very last move while Painter finds the play correct. In the first four moves Drawer reveals two edges v_1v_2 and v_3v_4.

Case 1: Painter uses only two colors, say $c_1 = c_3 = 1$ and $c_2 = c_4 = 2$.

Let v_5 be joined to v_1 and v_4. One may assume that Painter assigns $c_5 = 3$. Now v_6 joined with v_2, v_3 and v_5 is Drawer's winning move completing a copy of B.

Case 2: Painter uses three colors, say $c_1 = c_3 = 1$, $c_2 = 2$ and $c_4 = 3$.

Let v_5 be an isolated vertex. Then any color v_5 gets there are three vertices in distinct components and colored with 1, 2 and 3. Indeed, if $c_5 = 1$ then $c_2 = 2$, $c_4 = 3$, and if c_5 is different from 1, say $c_5 = 3$ then $c_1 = 1$, $c_3 = 2$.

Now v_6 joined to v_1, v_3 and v_5 is Drawer's winning move completing a copy of E.

Kits may behave differently, e.g. in section 2 we show that if K is the family of all forests, then $\chi^*(K) = max\{\chi^*(G) : G \in K\}$.

We just propose the following question: how large can be the on-line chromatic number of the family containing all graphs of on-line chromatic number k. This question is open even for $k = 3$.

The family of all $(P_2 + 2 \cdot P_1)$ - free graphs is an example, where FF is not effective (see [1]). Here we show that permutation graphs have the same property.

A graph G of order n is called a **permutation graph** if there exists a labeling L: $V(G) \leftrightarrow \{1, \ldots, n\}$ of its vertices and a permutation π of $\{1, \ldots, n\}$ such that $xy \in E(G)$ if and only if $(L(x) - L(y)) \cdot (\pi^{-1}(L(x)) - \pi^{-1}(L(y)) < 0$, where $\pi^{-1}(i)$ denotes the position of i in π.

A permutation graph is also characterized as a comparability graph such that its complement is also a comparability graph; a graph is called a comparability graph if there is a transitive orientation of its edges.

Note that triangle-free permutation graphs are bipartite, since comparability graphs contain no induced odd cycles of length greater than 3. On the other hand, since bipartite graphs are comparability graphs, a bipartite graph G is a permutation graph if and only if the edges of its complement has a transitive orientation.

Theorem 2. *FF is not effective on the family of permutation graphs.*

Proof. We show that for every integer k there exists a bipartite permutation graph G_k such that $\chi_{FF}(G_k) \geq k$. Define $G(n,k)$ with

$$V(G(n,k)) = \{a(1), \ldots, a(n)\} \cup \{b(1), \ldots, b(n)\} \text{ and}$$
$$E(G(n,k)) = \{a(i)b(j) : \quad 1 \leq i, j \leq n, \quad 1 \leq i - j < k\}.$$

Then $G(n,k)$ is a bipartite graph, and it is easy to check that its complement is ordered transitively in the following way:

$(a(i), a(j))$ and $(b(i), b(j))$ are arcs iff $i < j$,

$(a(i), b(j))$ is an arc iff $i \leq j$, and

$(b(j), a(i))$ is an arc iff $j \leq i - k$.

Let $G_k = G(k^2, k)$. For $p = 1, \dots, k$ let

$$Q_p = \cup \{a(ik+p), b(ik+p)\}.$$

Clearly Q_p is a stable (independent) set of G_k, $1 \leq p \leq k$. Observe that in the subgraph of G_k induced by $\cup\{Q_p : 1 \leq p \leq k\}$ $Q_1 \cup \cdots \cup Q_p$ is a maximal stable sequence decomposition. Indeed, for every $1 \leq p < q \leq k$ and $0 \leq j \leq k-q$ there is an edge from $a(jk+q)$ to $b(jk+p) \in Q_p$, and also from $b(jk+q)$ to $a((j+1)k+p) \in Q_p$. This maximal stable sequence partition with k classes has an extension to the whole G_k, whence FF colors G_k with at least k colors.

3. On-line chromatic number of trees

The first **canonical tree** T_1 is the one-point 'rooted' tree; for $k \geq 2$ the kth canonical tree T_k is the disjoint union of a 'left' and 'right' copy of T_{k-1} with an edge joining the two roots; the root of the left copy becomes the root of T_k.

We formulate some properties of canonical trees in the next lemmas. The first one proves easily by definition.

Lemma 3. Let x be the root of T_k. Then T_2 is an edge and for every $k \geq 3$

(C.1) $T_k - x$ has $k-1$ connected components isomorphic to $T_1, \dots, T_{k-1}$ and their root is joined to x.

(C.2) the right copy is T_{k-1} and the left copy is the subtree induced by x and $T_1 \cup \cdots \cup T_{k-2}$;

(C.3) each non-terminal vertex of T_k is adjacent to exactly one terminal vertex, and the deletion of the terminal vertices from T_k results in a tree isomorphic to T_{k-1}.

Lemma 4. Suppose that $K = \{T_k\}$ and N is a set of $k-1$ colors. Then Drawer has a strategy such that for arbitrary on-line coloring used by Painter the root of T_k will be colored with a color not in N.

Proof. The lemma is true for $k = 1$ with $N = \emptyset$ and for $k = 2$ with $|N| = 1$. Let $k \geq 3$ and assume that the lemma is true for T_i with a set $N_{i-1}, |N_{i-1}| = i-1$, whenever $1 \leq i \leq k-1$.

According to (C.1), T_k is the disjoint union of its root x and the component T_i with root x_i joined to $x, i = 1, \dots, k-1$. Drawer plays successively on the components

$T_1, \ldots, T_{k-1}$ and the last vertex he reveals will be the root x. Let N_{i-1} be the set of $i-1$ colors assigned to $x_1, \ldots x_{i-1}$. Then by induction, Drawer can obtain a color $c(x_i)$ not in N_{i-1}.

If $c(x_i)$ does not belong to N, then by (C.2), Drawer may freely exchange the role of the right copy T_i with the left copy in the $(i+1)$st canonical tree induced by $T_i, T_{i-1}, \ldots, T_1$ and x. Thus a color $c(x_i)$ not in N is obtained for the root of T_k.

Assume now that $c(x_i) \in N$ for every $i = 1, \ldots, k-1$. Since x is joined to each x_i, Painter has to assign a kth color to x which is obviously not in N.

Theorem 5. *If T is a tree and T_k is the kth canonical tree then*
(i)$\chi_A(T_k) \geq k$ for every on-line algorithm A;
(ii)$\chi_{FF}(T) \leq k$ if and only if T contains no induced subgraph isomorphic to T_{k+1};
(iii)$\chi^*(T) = \chi_{FF}(T)$.

Proof. (i) follows by Lemma 5.

(ii) If $T_k \subset T$, then by (i), $\chi_A(T) \geq \chi_A(T_k) \geq k$ for every on-line algorithm A. Sufficiency comes from the obvious fact that $\chi_{FF}(T) \geq k$ implies the existence of a copy of T_k in T.

(iii) Assume that the largest canonical tree contained in T is T_k. Then by (ii) and (i), $\chi_{FF}(T) \leq \chi_A(T_k) \leq \chi_A(T)$ follows proving that $\chi^*(T) = min\{\chi_A(T) : A \text{ on-line}\} \geq \chi_{FF}(T)$. The other direction is obvious.

As a colorally of (i) in Theorem 5 we obtain a result in [1]. For every integer k there exists a tree T such that $x_A(T) \geq k$ for every on-line algorithm A:

Proposition 6. On-line colorings are not effective on the family of all trees.

Note that (iii) says that in the case of trees no on-line algorithms can be more powerful than first fit. In (ii) of Theorem 5 trees are characterized with regard to on-line colorings. We formulate also the dual version in the following proposition.

Proposition 7. For a fixed forest H let K be the family of all H - free trees. Then $\chi_{FF}(T) \leq |V(H)| - 1$ for every $T \in K$.

Proof. Based on (C.3) it follows by induction on the order of H that T_{k+1} contains every forest of order $k+1$. Then the proposition follows from Theorem 5 with $k = |V(H)| - 1$.

Note that Proposition 7 is sharp as the example of a star shows in the role of H.

4. On-line chromatic number and forbidden forests

Proposition 8. Let F be a forest and assume that F is not an induced subgraph of the path P_5 and $K_{1,p} + K_1$ with $p \geq 1$. Then the first fit coloring is ineffective for the family of F - free graphs.

Proof. If F contains $2 \cdot K_1 + K_2$ as an induced subgraph, then $G = K_{n,n} - n \cdot K_2$ is an F - free graph with $\chi_{FF}(G) = n$, consequently, first fit is ineffective.

Thus it is enough to show that if F contains no $2 \cdot K_1 + K_2$ as an induced subgraph then F is an induced subgraph of P_5 or $K_{1,p} + K_1$.

Let the maximum path of F have k vertices. Then $k \leq 5$, since P_6 contains an induced $2 \cdot K_1 + K_2$. Moreover F is isomorphic to P_k whenever $k = 4$ or 5.

Now assume that $k \leq 3$, i.e., F is the disjoint union of stars. Let c_* be the number of non-trivial star components (i.e., those with more than one vertex), clearly $c_* \leq 2$. We can distinguish between the following cases:

if $c_* = 0$ then $F = p \cdot K_1$; if $c_* = 2$ then $F = 2 \cdot K_2$; if $c_* = 1$ then F has at most one trivial component. In each of these cases F is an induced subgraph of either P_5 or $K_{1,p} + K_1$.

Theorem 9. *If G is a $(K_{1,p} + K_1)$ - free graph $(p \geq 1)$, then $\chi_{FF}(G)$ is bounded in terms of $\omega(G)$.*

Proof.

Claim 1: $x(G) \leq f(\omega(G), p)$. We use induction on $\omega(G)$. The case $\omega(G) = 1$ is trivial. Assume that $\chi(G) \leq f(\omega(G), p)$ is true with $\omega(G) < t$ and let G be a $(K_{1,p} + K_1)$ - free graph with $\omega(G) = t$.

If all vertices of G has degree smaller than $R(t,p)$, then $\chi(G) \leq R(t,p)$, where $R(t,p)$ is the Ramsey function. Otherwise, there is a vertex of degree at least $R(t,p)$ in G and $K_{1,p} \subset G$ follows by definition of the Ramsey function. Fix this copy of $K_{1,p}$, then each vertex of $V(G) - V(K_{1,p})$ is adjacent to some vertex of $K_{1,p}$ because G is $(K_{1,p} + K_1)$ - free. Thus $V(G)$ can be covered by the vertices of at most $p+1$ stars and each star is at most $f(t-1,p)+1$ chromatic by the inductive hypothesis. Thus

$$f(t,p) \leq max\{R(t,p), (p+1)(f(t-1,p)+1)\}$$

and the claim is proved.

By Claim 1, $V(G)$ can be partitioned into m independent sets $A_1, \dots, A_m$, where $m \leq f(\omega(G), p)$. Assume that FF uses s colors to color G, let B_i denote the set of vertices colored by i $(i = 1, \dots, s)$. The *type* of B_i is defined to be the set of all indices j such that $B_i \cap A_j \neq \emptyset$.

Claim 2: if $B_i, i \in I \subset \{1, \dots, s\}$, have the same type, then

$$|I| \leq (m-1)(p-1) + 1 = h.$$

By renumbering for convenience, assume indirectly that $B_1, \dots, B_{h+1}$ are all of the same type. The first fit rule implies that any $x \in B_{h+1}$ is adjacent to some $y_i \in B_i$ for $i = 1, \dots, h$. By the pigeonhole principle, there exists $y_{i1}, \dots, y_{ip} \in A_j$

and $x, y_{i1}, \ldots, y_{ip}$ is an induced $K_{i,p}$ in G. Since the type of B_{h+1} is the same as types of $B_{i1}, \ldots, B_{ip}$, $B_{h+1} \cap A_j \neq \emptyset$.

Now selecting $z \in B_{h+1} \cap A_j$, z is not adjacent to $y_{i1}, \ldots, y_{ip}$ because A_j is independent, furthermore, z is not adjacent to x since $z, x \in B_{h+1}$ and B_{h+1} is independent. Therefore $\{x, y_{i1}, \ldots, y_{ip}, z\}$ induces a $K_{1,p}+K_1$ in G, and this contradiction proves Claim 2.

Then theorems follows since the number of types is bounded by $2^m - 1$ and Claim 2 gives $s \leq (2^m - 1)((m-1)(p-1)+1)$. Thus m is bounded in terms of $\chi(G)$ and p as proved in Claim 1.

Let F be a proper subforest of P_5. The effectiveness of first fit for F - free graphs is either trivial or is proved in [1]. The remaining case $F = P_5$ is considered now. We were not able to prove the effectiveness of FF in spite of several efforts was made so far. However, we have proved separately:

Theorem 10 [2]. *There is a function $f(\omega)$ such that $\chi^*(G) \leq f(\omega(G))$ for every P_5-free graph G.*

Theorem 10 says that on-line algorithms are effective on the family of all P_5-free graphs; and as it was proved in [1] on-line algorithms are ineffective on P_6-free graphs. This proposes the question of deciding about the status of on-line algorithms on the family of all F - free graphs for every tree F of radius 2 (different from P_5).

The case when F is the union of three edge disjoint copies of P_3 sharing a common endpoint has a particular interest. In fact if FF is effective on the family K of all these graphs, then since comparability graphs are contained in K, it follows that FF is effective on comparability graphs, a result that was proved in [3].

Our last result may show that on-line colorings are much more powerful than the FF algorithm.

Theorem 11. *Assume that $F_i, i = 1, \ldots, k$, are vertex disjoint forests. If on-line colorings are effective on the family of F_i - free graphs for $i = 1, \ldots, k$ then on-line algorithms are also effective on the family of all F - free graphs, where $F = \cup\{F_i : 1 \leq i \leq k\}$.*

Proof. It is clearly enough to prove the theorem for $k = 2$. Also we make a technical assumption that Painter knows the clique number, $\omega(G)$, of the graph being presented to him by Drawer. This assumption allows to present the proof as an iteration procedure of fixed depth. One can get rid of this assumption by rewriting the on-line algorithm in a recursive form.

Assume therefore that G is an $(F_1 \cup F_2)$ - free grah with $\omega(G) = t$ and Painter has on-line algorithm A_{t-1} to color any $(F_1 \cup F_2)$ - free graph H such that $\omega(H) = t - 1$.

Let A and B be on-line algorithm to color effectively any F_1 - free and F_2 - free graph, respectively.

Painter can use the following strategy to color G. Apply algorithm A until an induced copy of F_1 is found. If F_1 is never found, then A is the required algorithm. From this moment each vertex of G is classified into one of $n_1 + 1$ classes, where $V(F_1) = \{1, 2, \ldots, n_1\}$, as a follows. The current vertex x is put into class $i (i = 1, 2, \ldots, n_1)$ if xi is an edge of G (when x is adjacent to more than one vertex of F_1 the choice is abitrary). If x is not adjacent to any vertex of F_1, then x is put into class $n_1 + 1$. Let $C_1, C_2, \ldots, C_{n_1}$ be n_1 'copies' of algorithm A_{t-1}, each using a different set of colors. If x is the current vertex and x is from class $i, 1 \le i \le n_1$, then the color of x is given by C_i. If $i = n_1 + 1$, then x is colored by algorithm B. Clearly this is an effective on-line algorithm to color G. If $t = 1$, then G has no edge and the algorithm is trivial.

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