THE STRONG CHROMATIC INDEX OF GRAPHS

R. J. FAUDREE* and R. H. SCHELP**

Department of Mathematical Science

Memphis, Tennessee 38152

A. GYÁRFÁS and ZS. TUZA

Computer and Automation Institute of Hungarian Academy of Sciences

Abstract. Problems and results are presented concerning the strong chromatic index, where the strong chromatic index is the smallest k such that the edges of the graph can be k-colored with the property that each color class is an induced matching. This parameter was suggested by Erdös and Nesetril and it is related to an extremal problem of Bermond, Bond and Peyrat concerning induced matchings of graphs.

1. INTRODUCTION.

Graphs in this paper are finite, undirected, without loops, but parallel edges are allowed. A *strong matching* in a graph G is an *induced* subgraph of G that forms a matching (i.e. a set of pairwise disjoint edges of G, no two of them being adjacent to the same edge of G). The *strong chromatic index*, $sq(G)$, is the smallest integer k such that $E(G)$ (the edge set of G) can be partitioned into k strong matchings. For convenience we introduce *strong edge colorings*, i.e. edge colorings where each color class is a strong matching of the graph. Then $sq(G)$ is the smallest number of colors in a strong edge coloring of G. For simplicity, in this paper we use the term *coloring* for strong edge coloring. We address the following Vizing-type problem of Erdös and Nešetřil: give an upper bound for $sq(G)$ in terms of $\Delta(G)$, the maximum degree of G.

It is useful in studying $sq(G)$ to introduce some other parameters. Let $sm(G)$ denote the maximum number of edges in a strong matching of G. Also let $am(G)$ denote the maximum number of edges in G such that each pair of them are incident to some edge of G. Such an edge set is called an *antimatching*. Observe that the following obvious inequalities provide lower bounds for the strong chromatic index where $d(x)$ denotes the degree of vertex x, and $\sigma(G)$ is defined below.

$$sq(G) \geq am(G) \geq \max_{xy \in E(G)} \{d(x) + d(y) - 1\} := \sigma(G), \tag{1}$$

$$sq(G) \geq \frac{|E(G)|}{sm(G)}. \tag{2}$$

To obtain an upper bound on $sq(G)$, observe that the color of an edge $xy \in E(G)$ can be affected by the color of at most $2(\Delta(G) - 1)$ edges incident to xy and by the colors of at most $2(\Delta(G) - 1)^2$ "second neighbors" of xy. Therefore

$$sq(G) \leq 2\Delta^2(G) - 2\Delta(G) + 1 \tag{3}$$

1980 *Mathematics subject classifications* (1985 *Revision*): *Research partially supported under ORN Grant No. N00014-88-K- 0070. ** Research supported in part by NSA under Grant No. MDA 904-89-H-2026.

and a good coloring with at most $2\Delta^2(G) - 2\Delta(G) + 1$ colors can be found by the greedy algorithm.

The following constructions motivate the open problems. For even Δ, replace the vertices of a five-cycle by $\Delta/2$ vertices. For odd Δ, replace two consecutive vertices of a five cycle by $\frac{\Delta+1}{2}$ vertices and replace the three other vertices by $\frac{\Delta-1}{2}$ vertices ($\Delta \geq 3$). The graphs obtained are antimatchings with maximum degree Δ. For convenience, introduce the function

$$f(x) = \begin{cases} \frac{5}{4}x^2 & \text{if } x \text{ is even} \\ \frac{5}{4}x^2 - \frac{1}{2}x + \frac{1}{4} & \text{if } x \text{ is odd.} \end{cases}$$

Then, the antimatchings defined above have $f(\Delta)$ edges.

The main open problem is

CONJECTURE 1. $sq(G) \leq f(\Delta(G))$.

There are two interesting special cases of Conjecture 1.

CONJECTURE 2. $am(G) \leq f(\Delta(G))$.

CONJECTURE 3.. $|E(G)| \leq f(\Delta(G))sm(G)$.

The special case of Conjecture 3, when $sm(G) = 1$, has been asked by Bermond et al. in [1] and has been proved in [3]. A weaker form of Conjecture 2 is proved in this paper (Theorem 2).

If we restrict ourselves to bipartite graphs then the analogue of Conjecture 1 is the next conjecture.

CONJECTURE 4. *If G is bipartite then* $sq(G) \leq \Delta^2(G)$.

For bipartite graphs, the analogues of Conjectures 2 and 3 are true: $am(G) \leq \Delta^2(G)$ is proved in this paper (Theorem 1) and $|E(G)| \leq \Delta^2(G)sm(G)$ was proved in [4].

Most of our results concern the strong chromatic index of special graphs like trees, the cubes, Kneser graphs, and planar graphs. In this paper we also consider the relationship between the strong chromatic index and some extremal problems involving set systems. The paper concludes with several open problems concerning the strong chromatic index of graphs with maximum degree 3.

2. LARGEST ANTIMATCHINGS.

In the proofs that follow $\Gamma_F(x)$ denotes the set of vertices adjacent to x in a graph F and $d_F(x) = |\Gamma_F(x)|$.

Theorem 1. *If G is bipartite, then* $am(G) \leq \Delta^2(G)$.

Proof: Assume that X and Y are the two vertex classes of G and let G' be the subgraph of G induced by the edges of an antimatching of G. Select $xy \in E(G')$ such that $s = d_{G'}(x) = \Delta(G')$. Assume that $x \in X, y \in Y$. Set $X' = \Gamma_G(y)$, $Y' = \Gamma_G(x)$. Denote by $G[A, B]$ ($G'[A, B]$) the set of edges in G (in G') with one endvertex in A and with the other endvertex in B. Since the edges of G' define an antimatching in G we have

$$G'[X - X', Y - Y'] = \emptyset. \qquad \text{Moreover,}$$

at most $\Delta(G) - s$ edges of $G'[X' - x, Y - Y']$ can be incident to a vertex of $X' - x$. Using these properties and $|X'| \le \Delta(G)$, and $|Y'| \le \Delta(G)$ we obtain

$$\begin{aligned}|E(G')| &= |G'[X, Y']| + |G'[X' - x, Y - Y']| \\ &\le s|Y'| + (\Delta(G) - s)(|X'| - 1) \le s\Delta(G) + (\Delta(G) - s)(\Delta(G) - 1) \\ &= \Delta^2(G) - \Delta(G) + s \le \Delta^2(G). \ \blacksquare\end{aligned}$$

Theorem 2. *There is a constant $\epsilon > 0$ such that $am(G) \le (2 - \epsilon)\Delta^2(G)$.*

Proof: Suppose to the contrary that for every $\epsilon > 0$ there is an integer $\Delta = \Delta(\epsilon)$ and a graph G with maximum degree Δ that satisfies $am(G) > (2 - \epsilon)\Delta^2$. Fix a small ϵ and choose G accordingly. Let $A_0 \subseteq E(G)$ be an antimatching of size $am(G)$. We note that $\Delta(\epsilon) \to \infty$ when $\epsilon \to 0$, since $am(G) \le 2\Delta^2 - 2\Delta + 1$ by (3). Below, for convenience the ϵ_i will denote some functions of ϵ, such that $\epsilon_i \to 0$ as $\epsilon \to 0$.

Take an edge $e_0 = xy \in A_0$. Since every $e \in A_0$ meets the at most a $(2\Delta)-$ element vertex set $\Gamma(x) \cup \Gamma(y)$, and $|A_0| > (2-\epsilon)\Delta^2$, each of $\Gamma(x) \backslash (\Gamma(y) \cup \{y\})$ and $\Gamma(y) \backslash (\Gamma(x) \cup \{x\})$ contains at least $(1-\epsilon_1)\Delta$ vertices z with the following property: There are at least $(1-\epsilon_1)\Delta$ edges $zw \in A_0$ such that $w \notin \Gamma(x) \cup \Gamma(y)$. The set of these z is denoted by Z. Hence, $|Z| \ge (2 - 2\epsilon_1)\Delta$.

For $z \in Z$, set $F(z) = \{w \in V(G) \backslash (Z \cup e) : zw \in A_0\}$. By our assumptions, $\Delta \ge |F(z)| \ge (1 - \epsilon_1)\Delta$. Since A_0 is an antimatching, for each pair $z, z' \in Z, zz' \notin E(G)$ implies that the bipartite subgraph spanned between $F(z) \backslash \Gamma(z')$ and $F(z') \backslash \Gamma(z)$ is complete. Note that $|\Gamma(z) \backslash F(z)| \le \epsilon_1 \Delta$ for $z \in Z$.

Choose a $z_0 \in Z$, and let $\{z_1, \ldots, z_k\}$ be the set of vertices of Z not adjacent to z_0. Clearly $2\Delta \ge k \ge (2 - \epsilon_2)\Delta$. Letting $n_i = |F(z_0) \cap F(z_i)|$, we have a lower bound $m_i = \max\{0, (1 - 2\epsilon_1)\Delta - n_i\}$ for both $|F(z_0) \backslash \Gamma(z_1)|$ and $|F(z_1) \backslash \Gamma(z_0)|$. Thus, there are at least m_i^2 edges joining $F(z_i) \backslash \Gamma(z_0)$ to $F(z_0)$. Such an edge may belong to several $F(z_i)$; however, the degrees are bounded by Δ. Thus the set E_0 of edges having precisely one endpoint in $F(z_0)$ and being disjoint from Z satisfies

$$|E_0| \ge \frac{1}{\Delta} \sum_{i=1}^{k} m_i^2 \ge \frac{1}{\Delta k} \left(\sum_{i=1}^{k} m_i \right)^2 . \tag{4}$$

On the other hand, we have at least

$$\sum_{i=1}^{k} n_i$$

edges with one endpoint in $F(z_0)$ and the other in Z. Therefore, again by the degree assumption and the lower bound for k,

$$\begin{aligned}|E_0| &\le \Delta^2 - \sum_{i=1}^{k} n_i \\ &\le \Delta^2 + \sum_{i=1}^{k} (m_i - (1 - 2\epsilon_1)\Delta) \\ &= (\sum_{i=1}^{k} m_i) - (1 - \epsilon_3)\Delta^2. \end{aligned} \tag{5}$$

Rearrangement of (4) and (5) yields

$$(1-\epsilon_3)\Delta^2 \leq \left(\sum_{i=1}^{k} m_i\right)\left(1-\frac{\sum_{i=1}^{k} m_i}{\Delta k}\right) \leq \frac{\Delta k}{4} \leq \frac{1}{2}\Delta^2,$$

a contradiction because $\epsilon_3 < 1/2$ holds for a sufficiently small ϵ. ∎

We finish this section by showing that $am(G)$ and $sq(G)$ can be of different orders of magnitude for an infinite sequence of graphs G. Let P_q be a projective plane of order q. The bipartite graph $B(P_q)$ is defined by letting the points and lines of P_q be the vertex classes of $B(P_q)$ and letting adjacency be defined by the incidences of P_q. Illés and Szönyi proved [5] that at most $q\sqrt{q}+1$ lines of P_q can be strongly represented (and for certain planes this is best possible). In our terminology their theorem says that $sm(B(P_q)) \leq q\sqrt{q}+1$. Using (2), this implies

$$sq(B(P_q)) \geq \frac{(q^2+q+1)(q+1)}{q\sqrt{q}+1}$$

Therefore $B(P_q)$ is an example of a C_4-free graph G with strong chromatic index at least $\Delta(G)\sqrt{\Delta(G)}$. On the other hand, it is easy to see that $am(G) \leq 2\Delta(G)-1$ for any C_4-free bipartite graph.

3. CHROMATIC INDEX OF SPECIAL GRAPHS.

Theorem 3. *If G is a tree then*

$$sq(G) = \sigma(G)$$

Proof: It is obvious that $sq(G)$ is at least the stated value (see inequality (1)). The other direction is proved by induction on $|V(G)|$. Let A denote the set of vertices of degree one in G. Let uv be an edge of the tree T spanned by $V(G)-A$ in G such that u is of degree one in T. (If $|v(T)|=1$ then G is a star, a trivial case.) Select $w \in A$ such that $uw \in E(G)$. Applying the induction hypothesis for the tree $G-w$, the edges of $G-w$ can be colored by at most $\sigma(G-w)$ colors. From the choice of u and w, the uncolored edge $uw \in E(G)$ has at most $d_{G-w}(u)+d_{G-w}(v)-1 < d_G(u)+d_G(v)-1 \leq \sigma(G)$ forbidden colors. Therefore a "free" color can be assigned to $uw \in E(G)$. ∎

Theorem 4. *For the d-dimensional cube Q^d,*

$$sq(Q^d) = am(Q^d) = 2d \quad \textit{if} \quad d \geq 2.$$

Proof: Notice that any C_4 in Q^d together with the edges incident to two consecutive vertices of the C_4 give a subgraph of Q^d with $2d$ edges and this subgraph is an antimatching. Thus $sq(Q^d) \geq am(Q^d) \geq 2d$.

To prove the upper bound, represent a vertex x of Q^d by a 0-1 vector $v(x)$ of length d. The two vertices x,y are adjacent if and only if $v(x)$ and $v(y)$ are equal in all but one coordinate. Define the i-th edge class E_i as the set of edges xy in which $v(x)$ and $v(y)$ differ in the i-th coordinate $(1 \leq i \leq d)$. A refinement $E_i^1 \cup E_i^2 = E_i$ of this edge partition is obtained in the following way: an edge $xy \in E_i$ belongs to $E_i^j (1 \leq i \leq d,\ 1 \leq j \leq 2)$ if and only if the sum of all coordinates of $v(x)$ (or $v(y)$) except for the i-th one, is congruent to j (mod 2). Obviously, each E_i^j is a strong matching in Q^d, implying $sq(Q^d) \leq 2d$. ∎

The following bipartite graph, RD^d, is often referred to as the "*revolving door*" graph: the vertices of RD^d are the $(d-1)$-element subsets and d-element subsets of a $(2d-1)$-element ground set. Two vertices are adjacent if one of the corresponding sets is a subset of the other.

Theorem 5. *For the revolving door graph RD^d,*

$$sq(RD^d) = am(RD^d) = 2d - 1 = \sigma(RD^d)$$

Proof: Since RD^d is a d-regular graph, any edge together with its incident edges defines an antimatching of $2d$-1 edges. Thus $sq(RD^d) \geq am(RD^d) \geq 2d-1$. Therefore the theorem follows by giving a good $(2d-1)$-coloring of RD^d. To see this, let $X = \{1, 2, \ldots, 2d-1\}$ be the ground set, and select $A \subset B \subset X$ such that $|A| = d-1, |B| = d$. The edge of RD^d corresponding to the pair (A, B) is assigned the color identified with the set $B - A$. It is easy to check that this is a good coloring of $E(RD^d)$ using $2d-1$ colors. ▮

The next result concerns a rather special family of graphs, for which Conjecture 4 is true. Although Conjecture 4 holds for these graphs probably the best upper bound is much smaller, in fact linear in $\Delta(G)$.

Theorem 6. *If G is a graph in which all its cycle lengths are divisible by 4, then $sq(G) \leq \Delta^2(G)$.*

Proof: The proof is by induction on $|E(G)|$. If $d(x) = 1$ for some $xy \in E(G)$, then delete xy from G. Then a coloring of $G - xy$ can be extended to G, since at most $(\Delta(G)-1)\Delta(G)$ colors are forbidden on xy. Therefore the minimum degree of G is at least 2. Select a path $x_1, x_2, \ldots, x_t$ of maximum length in G. There is an x_s such that $s \neq t-1$ and $x_s x_t \in E(G)$. Now $d(x_t) = 2$ follows, since G has no cycle with a diagonal. Also, $d(x_{s+1}) = 2$ from the maximality of the path length. Delete the two edges ($x_t x_{t-1}$ and $x_t x_s$) incident to x_t and color the remaining graph with at most $\Delta^2(G)$ colors. There are at most $\Delta(\Delta-1)+\Delta-1 = \Delta^2 - 1$ forbidden colors for the edge $x_t x_{t-1}$. Moreover, there are at most $\Delta(\Delta-1)+1$ forbidden colors in $G - x_t x_s - x_t x_{t-1}$ for the edge $x_t x_s$, since $d(x_{s+1}) = 2$. Therefore $x_t x_{t-1}$ can be colored with a free color and there is an additional free color for the edge $x_t x_s$, if $\Delta(G) \geq 3$. If $\Delta(G) = 2$, then G is the union of disjoint cycles and 4 colors are clearly sufficient for a good coloring. ▮

The Kneser graph KN_n^m is defined for $m \geq 2n$ as follows. The vertices are the n-element subsets of a fixed m-element ground set. Two vertices are adjacent if and only if the corresponding sets are disjoint. The set of edges defined by pairs of disjoint n-sets in a fixed $2n$-element subset is clearly a strong matching. Since these $\binom{m}{2n}$ strong matchings cover all edges of KN_n^m, we immediately obtain

$$sq(KN_n^m) \leq \binom{m}{2n}. \tag{4}$$

We give two simple proofs to show that equality holds here. The first proof is based on the following proposition which also appears in [6] with the same proof.

Proposition 7.

$$sm(KN_n^m) = \binom{2n}{n}/2$$

Proof: A strong matching in KN_n^m corresponds to a system of pairs $(A_i, B_i), i = 1, 2, \ldots, k$ such that $A_i \cap B_j, A_i \cap A_j, B_i \cap B_j$ are non-empty sets, if $i \neq j$ but $A_i \cap B_i = \emptyset$ for all i ($|A_i| = |B_i| = n$ for all i). Define $A_{i+k} = B_i$, $B_{i+k} = A_i$ for $i = 1, 2, \ldots, k$. Then $\{(A_i, B_i) : i = 1, 2, \ldots, 2k\}$ is a system of pairs such that $A_i \cap B_j \neq \emptyset$ for $1 \leq i, j \leq k$, $i \neq j$ and $A_i \cap B_i = \emptyset$ for $1 \leq i \leq 2k$. Applying a theorem of Bollobás ([2]),

$$2k \leq \binom{2n}{n} \quad \text{follows. ▮}$$

Theorem 8.

$$sq(KN_n^m) = \binom{m}{2n}$$

Proof:

The upper bound is given in (4). Since KN_n^m has $\frac{1}{2}\binom{m}{n}\binom{m-n}{n}$ edges inequality (2) yields

$$sq(KN_n^m) \geq \left(\frac{1}{2}\binom{m}{n}\binom{m-n}{n} \Big/ \frac{1}{2}\binom{2n}{n}\right) = \binom{m}{2n}$$

applying Proposition 7. ∎

The second proof that

$$sq(KN_n^m) \geq \binom{m}{2n}$$

comes from the following statement.

Proposition 9.

$$am(KN_n^m) \geq \binom{m}{2n}$$

Proof: Consider an ordering of the m-element ground set X. For each $2n$-element subset H of X let A_H denote the first n elements of H under the given ordering, and set $B_H = H - A_H$. It is easy to check that the edges in KN_n^m corresponding to the pairs (A_H, B_H) form an antimatching. ∎

The first part of (1) and Proposition 9 imply Theorem 8.

Theorem 10. *If G is a planar graph, then $sq(G) \leq 4\Delta(G)+4$. Moreover for every integer ≥ 2 there exists a planar graph G with $\Delta(G) = \Delta$ and $sq(G) = 4\Delta - 4$.*

Proof: The second part of the theorem follows by identifying a fixed C_4 in $K_{2,m}$ with a fixed C_4 in $K_{m,2}$ by glueing together corresponding edges of the C_4's. The graph obtained is a planar antimatching with maximum degree m and it has $4m - 4$ edges.

To see the first part, apply Vizing's theorem to decompose $E(G)$ into at most $\Delta(G)+1$ (not necessarily strong) matchings. If M is one of these matchings, define the graph $G(M)$ as one with vertex set M and two vertices in $G(M)$ adjacent if and only if the corresponding two edges in M do not form a strong matching in G. Since $G(M)$ is planar, in fact can be obtained from the induced subgraph $G|_M$ by contracting the edges of M, its chromatic number is at most 4 by the four color theorem. Therefore M can be decomposed into 4 strong matchings. ∎

We conclude this section by giving a small list of regular graphs with their parameters related to strong matchings.

G	$\Delta(G)$	$sq(G)$	$am(G)$	$sm(G)$
Petersen graph	3	5	5	3
Heawood graph	3	7	5	3
Dodecahedron	3	5	5	6
Octahedron	4	12	12	1
Icosahedron	5	15	13	2

4. OPEN PROBLEMS FOR GRAPHS WITH $\Delta(G) = 3$.

In each of the following problems we assume that $\Delta(G) = 3$.

(1) The strong chromatic index $sq(G) \leq 10$. (This special case of Conjecture 1 is true if G is not 3-regular.)
(2) If G is bipartite, then $sq(G) \leq 9$. (This is a special case of Conjecture 4.)
(3) If G is planar, then $sq(G) \leq 9$. (The complement of C_6 shows that if true, this is best possible.)
(4) If G is bipartite and if for each edge $xy \in E(G), d(x) + d(y) \leq 5$, then $sq(G) \leq 6$. (If true, this is best possible. If the bipartite condition is dropped, then $sq(G) \leq 7$ follows and this is best possible.)
(5) If G is bipartite and $C_4 \not\subset G$, then $sq(G) \leq 7$.
(6) If G is bipartite and its girth is large, then $sq(G) \leq 5$

REFERENCES

1. C. Bermond, F. Bond, M. Paoli, C. Peyrat, *Surveys in Combinatorics*, London Math. Soc. Lecture Notes **82** (1983), 1–30.
2. B. Bollobás, *On generalized graphs*, Acta Math. Acad. Sci. Hungar, **16** (1965), 447–452.
3. F. R. K. Chung, A. Gyárfás, W. T. Totter, Zs. Tuza, *The maximum number of edges in $2K_2$-free graphs of bounded degree*, to appear in Discrete Math..
4. R. J. Faudree, A. Gyárfás, R. H. Schelp, Zs. Tuza, *Induced matchings in Bipartite Graphs*, to appear in Discrete Math.
5. T. Illés, T. Szönyi, *Private communication.*
6. T. Lehel, *Tree structures, packing and covering in hypergraphs, Thesis (1982)*, [in Hungarian].
7. Zs. Tuza, *The method of set-pairs for extremal problems in hypergraphs*, "Finite and Infinite Sets", Proc. Colloq. Math. Soc. János Bolyai, Eger (Hungary) 1981, North-Holland, 749–762.