MONOCHROMATIC COVERINGS IN COLORED COMPLETE GRAPHS

P. ERDÖS

Hungarian Academy of Sciences

R. J. FAUDREE*

Memphis State University

R. J. GOULD*

Emory University

A. GYÁRFÁS

Hungarian Academy of Sciences

C. ROUSSEAU

Memphis State University

R. H. SCHELP

Memphis State University

Abstract. We consider the following question: For a fixed positive integer t and a fixed r-coloring of the edges of K_n, what is the largest subset B of $V(K_n)$ monochromatically covered by some t element subset of $V(K_n)$?

1. INTRODUCTION.

Let G be a graph, $A, B \subseteq V(G)$. The set A is said to *cover* (or *dominate*) B if for every $y \in B - A$ there exists an $x \in A$ such that $xy \in E(G)$. Thus if A covers B then A covers $A \cup B$. In what follows this idea of covering will be applied to the monochromatically colored subgraphs of K_n obtained by coloring each of its edges by one of a fixed set of colors.

A problem of this type due to Erdös and Hajnal is given in the following conjecture.

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CONJECTURE. (ERDÖS, HAJNAL). *For given positive integers n, t and any 2-coloring of the edges of K_n there exists a set $X_t \subseteq V(K_n)$, with at most t vertices, which monochromatically covers at least $(1 - 1/2^t)n$ of the vertices of K_n.*

This conjecture is trivially true for $t = 1$, was proved by Erdös and Hajnal for $t = 2$, and proved in more general form in [1]. Before stating this general form we introduce additional terminology. If the edges of a graph have been 2-colored, we assume the colors are red and blue, and refer to a covering in the resulting red(blue) subgraph as an *r-covering* (*b-covering*). The result proved in [1] is the following.

THEOREM A. [1]. *Let $G = [X, Y]$ be a 2-colored complete bipartite graph, t be a non-negative integer, and β any real number satisfying $0 < \beta < 1$. Then at least one of the following two statements is true.*

(1) *Some set of t vertices of X r-covers all but at most $\beta^{t+1}(|X| + |Y|)$ vertices of Y.*

(2) *Some set of t vertices of Y b- covers all but at most $(1 - \beta)^{t+1}(|X| + |Y|)$ vertices of X.*

This gives as an immediate corollary the following generalization of the Erdös - Hajnal conjecture. (The case when $\beta = 1/2$ is the Erdös - Hajnal conjecture.)

COROLLARY [1]. *Let the edges of K_n be 2-colored, p a fixed vertex of K_n, k a positive integer, and $\beta \in (0, 1)$. Then there exists a set $A \subseteq V(K_n)$ such that $p \in A, |A| \le k$, and A either r-covers at least $(1 - \beta^k)n$ vertices of K_n or b-covers at least $[1 - (1 - \beta)^k]n$ vertices of K_n.*

The proof of Theorem A given in [1] is constructive. In fact a greedy low order polynomial algorithm will find the covering set. Thus one might feel that the result of Theorem A is not sharp, but this is not the case as is shown by the next result.

THEOREM B [1]. *For any fixed $\epsilon > 0$ and positive integer t there exists an $n_0 = n_0(\epsilon, t)$ and a 2-coloring of the edges of K_n for $n \geq n_0$ such that each t-element subset fails to monochromatically cover at least $(1/2^t - \epsilon)n$ vertices of K_n.*

This leaves as unsettled the general question of what happens if r-colorings of the edges of K_n are considered instead of 2-colorings. In particular if the edges of K_n are r-colored, then for which t does there exist some set of t vertices which monochromatically covers at least $(1 - (1 - 1/r)^t)n$ vertices of K_n?

No such result can hold for arbitrary r and t, not even when $r = 3$ and $t = 3$. This was first noticed by H. A. Kierstead who gave the following example. Three color the edges of K_n by partitioning its vertex set into three sets A_1, A_2, A_3 of equal order. If $1 \leq i \leq j \leq 3$ and $x \in A_i, y \in A_j$, then color edge xy with color i. Clearly any three vertices monochromatically cover at most $2n/3$ vertices of K_n, while in this case $(1-(1-1/r)^t)n = 19n/27$. We shall see in the next section that this generalization will essentially hold for many values of r and t. Also we shall show when $r = 3$ that the expected number of vertices monochromatically covered by a "small" set is $2n/3$.

RESULTS (many colors).

The example of Kierstead shows no "small set" of vertices can be found which, in general, monochromatically covers substantially more than $2n/3$ vertices of K_n under a 3-coloring of its edges. The first result of the paper shows that a covering of $2n/3$ vertices can be realized using a "small set" of vertices.

THEOREM 1. *Three color the edges of K_n. Then there exists a set of at most k vertices in $K_n (k \leq 22)$ which monochromatically covers at least $2n/3$ of its vertices.*

The upper bound of 22 on k is only a consequence of the method of proof of the theorem. A random 3-coloring of $E(K_n)$, with each color of equal probability, provides an example of a 3-colored graph in which each pair of vertices monochromatically covers at most $5n/9$ vertices. Thus we know $3 \leq k \leq 22$. Most likely $k = 3$ will suffice, but presently we have no proof.

We next consider the general question mentioned earlier; for which r, t does there exist a t element set which monochromotically covers at least $(1 - (1 - 1/r)^t)n$ vertices for any r-coloring of $E(K_n)$? With this in mind we prove the next theorem.

THEOREM 2. *Let G be a graph on n vertices and $cn^2/2$ edges, $0 < c < 1$, and let t be a fixed positive integer. Set $\Delta(G) = \Delta n, N_1 = \Delta$, and define N_t recursively by $N_t = c + (1 - \Delta)N_{t-1}$. Then there exists t vertices of G which cover at least $(\max\{\Delta, N_t\})n$ vertices of G. Furthermore $\max\{\Delta, N_t\} \geq \min\{1 - (1 - c)^t, \sqrt{c}\}$.*

One should observe that if G is a regular graph, then $\Delta = c$ and $N_t = c + (1 - c)N_{t-1} = 1 - (1 - c)^t$, while if $G = K_{\sqrt{c}n}$, then $\Delta = \sqrt{c}$ and $N_t = c + (1 - \sqrt{c})N_{t-1} = \sqrt{c}$.

It can be checked that the following slight modification of Theorem 2 is also true. Let the n vertex graph G have $c(n - 1)^2/2$ edges and set $\Delta(G) = \Delta(n - 1)$. Then (with t and N_t as defined) G contains a t element set which covers at least $(\max\{\Delta, N_t\})(n - 1)$ vertices of G. The next corollary is a consequence of this modified from of Theorem 2 and gives a partial answer to the question asked earlier.

COROLLARY 1. *Let t be a fixed positive integer and let r be fixed and large. If the edges of K_n are r colored and n is large with respect to r, then there exists t vertices of K_n which monochromatically cover at least $(1 - (1 - 1/r)^t)(n - 1)$ of its vertices.*

PROOF: The dominant color class in the colored K_n has at least $(n^2 - n)/(2r) \geq (1/r)(n - 1)^2/2$ edges. Setting $c = 1/r$ choose r large enough such that, for all large n, $(1 - (1 - c)^t \leq \sqrt{c}$ and apply the modified version of Theorem 2. ∎

COROLLARY 2. *Let K_n be edge colored with r colors and let t be a fixed positive integer. If either $t = 2$ or if the color class with the majority of edges is a regular graph, then there exists t vertices of $V(K_n)$ which monochromatically cover at least $(1 - (1 - 1/r)^t)(n - 1)$*

of its vertices.

PROOF: If $r = 2$ and $t = 2$ the result follows from the corollary of Theorem A, while if $r \geq 3$ and $t = 2$ the result follows from the modified version of the theorem, since

$$1 - (1-c)^2 \leq \sqrt{c} \quad \text{for} \quad 0 \leq c \leq \frac{1}{3}.$$

If the color class with the majority of edges is regular, then that colored graph has at least $c(n-1)^2/2$ edges with $c \geq 1/r$ so that $N_t \geq 1 - (1-1/r)^t$. Hence the modified version of Theorem 2 again applies. ∎

PROOFS OF THEOREMS 1 AND 2.

Proof of Theorem 1:

Assume that the three colors with which $E(K_n)$ has been colored are named 1, 2, and 3. Throughout the proof we use the following notation. For $B \subseteq V(K_n)$ and $x \in V(K_n)$ let $d_i(x)$ denote the degree of x in the subgraph of K_n induced by color i, and let $d_i^{(B)}(x)$ be its degree relative to the set B.

The proof is indirect, so we suppose the Theorem is false. For each $i (1 \leq i \leq 3)$ select a set A_i of vertices that is covered by k vertices, let $B_i = V(K_n) - A_i$. Choose A_i such that the maximum degree in color i with respect to B_i is $\delta_i n$, a minimum.

Assume without loss of generality that $\delta_1 \leq \delta_2 \leq \delta_3$. Further, since $|A_i| < 2n/3$ by assumption, $|B_i| = (1/3 + \epsilon_i)n$ where $\epsilon_i > 0$. Let $C_i = \{z \epsilon B_i | d_i(z) \geq n/6\}$.

Since $\sum_{x \epsilon V(K_n)} d_i^{(B_i)}(x) \leq \delta_i n^2, |C_i| \leq \delta_i n^2/(n/6) = 6\delta_i n$. If $y \in (B_1 \cap B_2) - (C_1 \cup C_2)$, then $d_1(y) < n/6$ and $d_2(y) < n/6$, so that $d_3(y) \geq 2n/3$. Since this is impossible, $|B_1 \cap B_2| \leq |C_1 \cup C_2| \leq 6(\delta_1 + \delta_2)n$.

Next observe

$$\sum_{x \epsilon B_2} d_2^{(B_1)}(x) = \sum_{x \epsilon B_1} d_2^{(B_2)}(x) \leq (1/3 + \epsilon_1)\delta_2 n^2$$

and

$$\sum_{x\epsilon B_1} d_1^{(B_2)}(x) = \sum_{x\epsilon B_2} d_1^{(B_1)}(x) \leq (1/3+\epsilon_2)\delta_1 n^2.$$

Thus there exists a $z_1 \epsilon B_1$ such that

$$d_3^{(B_2)}(z_1) \geq (1/3+\epsilon_2)n - ([(1/3+\epsilon_1)\delta_2 + (1/3+\epsilon_2)\delta_1]/[1/3+\epsilon_1])n,$$

and a vertex $z_2 \epsilon B_2$ such that

$$d_3^{(B_1)}(z_2) \geq (1/3+\epsilon_1)n - ([(1/3+\epsilon_1)\delta_2 + (1/3+\epsilon_2)\delta_1]/[1/3+\epsilon_2])n.$$

Therefore $\{z_1, z_2\}$ covers in $B_1 \cup B_2$(in color 3) at least αn vertices, where

$$\begin{aligned} \alpha = 2/3 + (\epsilon_1+\epsilon_2) - 6(\delta_1+\delta_2) - [1/3+\epsilon_1)\delta_2 + (1/3+\epsilon_2)\delta_1] \Big/ [1/3+\epsilon] \\ -[(1/3+\epsilon_1)\delta_2 + (1/3+\epsilon_2)\delta_1] \Big/ [1/3+\epsilon_2] \\ \geq 2/3+\epsilon_1+\epsilon_2 - 10(\delta_1+\delta_2). \end{aligned} \tag{1}$$

Note that this later expression implies $\delta_1 + \delta_2 > 0$, since it has been assumed that the theorem is false.

Next select vertices $z_3, z_4, \ldots, z_k$ such that $\{z_1, z_2, z_3, \ldots, z_k\}$ covers as many vertices in K_n as possible in color 3. Since no such set covers $2n/3$ vertices, it follows from (1) that at least one of the vertices in the set $\{z_3, z_4, \ldots, z_k\}$ covers (in color 3) at most $([10(\delta_1+\delta_2) - (\epsilon_1+\epsilon_2)]/[k-2])n$ vertices not covered by the remaining ones. Hence if A_3 is the set covered in color 3 by $\{z_1, z_2, \ldots, z_k\}$ and $B_3 = V(K_n) - A_3$, then for $x \epsilon V(K_n)$

$$d_3^{(B_3)}(x) \leq ([10(\delta_1+\delta_2) - (\epsilon_1+\epsilon_2)]/[k-2])n.$$

Therefore $\delta_3 \leq [10(\delta_1+\delta_2) - (\epsilon_1+\epsilon_2)]/[k-2]$. But then $0 < (\delta_1+\delta_2)/2 \leq \delta_3 \leq 10(\delta_1 + \delta_2)/(k-2)$, a contradiction for $k > 22$. ∎

Before presenting the proof of Theorem 2 we prove the following needed polynomial inequality.

LEMMA 1. *For $k > 1$ let P_t be the polynomial of degree $t+1$ given by*

$$P_t(x) = [1-(1-c)^t]x + (c-x^2)(1-x)^{t-1},$$

where $0 < c < 1$ satisfies $1-(1-c)^t < \sqrt{c}$. Then $P_t(x) \le c$ for $c \le x \le \sqrt{c}$.

PROOF: The equation $1-(1-c)^t = \sqrt{c}$ has a unique solution $c = c_t$ in the open interval $(0,1)$ with $1-(1-c)^t < \sqrt{c}$ if and only if $c \epsilon (0, c_t)$. It can be shown that $1-(1-c)^t > \sqrt{c}$ for $c = 1/(t^2-t)$ and $1-(1-c)^t < \sqrt{c}$ for $c = 1/(t^2-t+1)$, so that

$$(2) \qquad 1/(t^2-t+1) < c_t < 1/(t^2-t).$$

Since $P_t(c) = c$ and $P_t(\sqrt{c}) = [1-(1-c)^t]\sqrt{c} < c$, to show $P_t(x) \le c$ for $c \le x \le \sqrt{c}$, it suffices to prove P_t has no relative maximum in $(c, \sqrt{c})$.

We suppose P_t has a relative maximum in $(c, \sqrt{c})$ and show this leads to a contradiction. For $t = 2$ this is easy to check, so we assume $t \ge 3$. Suppose $x_1 \epsilon (c, \sqrt{c})$ satisfies $P_t'(x_1) = 0$ and $P_t''(x_1) < 0$. Using (2) one can check that $P_t'(c) = 1-(1-c)^{t-1}[t(t+1)c-2] < 0$ and $P_t''(c) = (1-c)^{t-2}[t(t+1)c-2] < 0$. It follows that there is a point $x_0 \epsilon (c, x_1)$ such that $P_t'(x_0)$ and $P_t''(x_0) > 0$, so that P_t'' must have one zero in (c, x_0) and another in (x_0, x_1). But $P_t''(x) = (1-x)^{t-3}[(t-1)(t-2)c - 2 + 4tx - t(t+1)x^2]$ so that the sum of these zeros is $4/(t+1)$. Thus $4/(t+1) < 2\sqrt{c}$ so that from (2)

$$4/(t+1)^2 < c < 1/(t^2-t),$$

which is impossible for $t \ge 3$. ∎

The recursive definition of N_t given in Theorem 2 is such that $N_t = N_t(\Delta) = c[1-(1-\Delta)^{t-1}]/\Delta + (1-\Delta)^{t-1}\Delta$. Therefore P_t as defined in Lemma 1 satisfies

$$P_t(\Delta) = (1-(1-c)^t)\Delta + c - \Delta N_t(\Delta).$$

Hence $P_t(\Delta) \leq c$ if and only if $N_t(\Delta) \geq [1-(1-c)^t]$. This means that under the conditions of Lemma 1, when $t > 1$ and $0 < c < 1$ satisfies $1 - (1-c)^t < \sqrt{c}$,

$$\text{then} \quad N_t(\Delta) \geq 1 - (1-c)^t \quad \text{for} \quad c \leq \Delta \leq \sqrt{c}.$$

One can also show by straightforward calculations that for $c \leq \Delta \leq \sqrt{c}$ and $1-(1-c)^t \geq \sqrt{c}$ that $N_t(\Delta) \geq \sqrt{c}$. Thus for all $c \leq \Delta \leq \sqrt{c}$

$$(3) \qquad N_t(\Delta) \geq \min\{1-(1-c)^t, \sqrt{c}\}.$$

Proof of Theorem 2:

We prove by induction on t that there exists t vertices which cover at least $(\max\{\Delta, N_t\})n$ vertices of G. This is clear for $t = 1$, so we assume the result for $t - 1$.

Let A denote the set of largest order covered by a $t-1$ vertex set of G and let $B = V(G) - A$. Choose δ such that $(N_{t-1} + \delta)n = |A|$ and $|B| = (1 - N_{t-1} - \delta)n$. Further choose the smallest l such that the degree $d^{(B)}(x)$ of each vertex x of G, relative to set B, is at most ln. Thus the maximum number N of edges in G (counting edges in A, from A to B, and in B) is at most

$$N = [(N_{t-1} + \delta)(\Delta - l)n^2]/2 + (N_{t-1} + \delta)ln^2 + [(1 - N_{t-1} - \delta)ln^2]/2.$$

Hence $cn^2/2 \leq N$ which is equivalent to $l \geq c - N_{t-1}\Delta - \delta\Delta$.

Since by assumption $t-1$ vertices of G cover $(N_{t-1} + \delta)n$ vertices, t vertices cover $(\max\{\Delta, l+N_{t-1}+\delta\})n$ vertices of G, where $l \geq c - N_{t-1}\Delta - \delta\Delta$. Hence there exist t vertices of G which cover at least $(\max\{\Delta, l+N_{t-1}+\delta\})n \geq (\max\{\Delta, c+(1-\Delta)N_{t-1}+\delta(1-\Delta)\})n$ vertices of G. Since this last expression is minimum for $\delta = 0$, it follows that there exist t vertices of G which cover at least $(\max\{\Delta, N_t\})n$ vertices, where $N_t = c + (1-\Delta)N_{t-1}$.

The fact that $\max\{\Delta, N_t\} \geq \min\{1-(1-c)^t, \sqrt{c}\}$ follows from (3) and from the fact that the maximum degree of G is at least cn. ∎

4. CONCLUDING REMARKS.

It would be nice to improve the result of Theorem 1 and show that if $E(K_n)$ is 3-colored, then there exist a 3-vertex set that monochromatically covers at least $2n/3$ vertices. Some evidence for this possibility is provided by Theorem 2. Note that by Theorem 2 there exists 3 vertices which monochromatically covers at least $n/\sqrt{3}$ of the vertices. Also by Theorem 2 a 3-colored K_n with as many as $(4/9)(n^2/2)$ edges in one color has 2 vertices which monochromatically covers $2n/3$ vertices. Thus it appears that Theorem 1 may hold for $k = 3$.

Also a theorem parallel to that of Theorem 1 could be considered for r colors, where $r \geq 4$. To consider this parallel problem we modify the definition of cover to say that A covers B in G if each vertex of $B - A$ is adjacent to a vertex of A and each vertex of A is incident to an edge of G. The vertices in A can not be isolated. With this alternate definition, define $f(r)$ as the largest real number such that for all n and all r-colorings of $E(K_n)$ at least $f(r)n$ vertices can be monochromatically covered by some set $A \subseteq V(K_n)$.

In [2] this problem is considered in a different setting. It is shown there that $f(r)$ satisfies the following tabular results.

r	2	3	4	5	6	7	8	9	10	11	12	13
$f(r)$	1	2/3	3/5	5/9	1/2	3/7	5/12	2/5	3/8	5/14	1/3	4/13

Thus, for example, if $E(K_n)$ is 4-colored, then does there exist a "small set" which monochromatically covers at least $3n/5$ vertices?

An additional problem is to find the order of the smallest set which monochromatically covers $f(r)n$ vertices in any r- coloring of $E(K_n)$. This order was shown in Theorem 1 to be at most 22 for $r = 3$ and conjectured to be 3 for $r = 3$.

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