Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B
All Rights Reserved by Academic Press, New York and London

Vol. 51, No. 1, January 1991
Printed in Belgium

Vertex Coverings by Monochromatic Cycles and Trees

P. ERDŐS, A. GYÁRFÁS,* AND L. PYBER†

Computer and Automation Institution, Hungarian Academy of Science, Budapest, XI, Kende u., 13–17, Hungary

Communicated by the Managing Editors

Received September 30, 1987

If the edges of a finite complete graph K are colored with r colors then the vertex set of K can be covered by at most $cr^2 \log r$ vertex disjoint monochromatic cycles. Several related problems are discussed.

1. INTRODUCTION

Assume that K is a finite complete graph whose edges are colored with r colors ($r \geqslant 2$). How many monochromatic paths (or cycles) are needed to cover (or partition) the vertex set of K? Throughout the paper single vertices and edges are considered to be cycles. It is not obvious that these numbers depend only on r. The following conjecture is from [12].

If the edges of a (finite undirected) complete graph K are colored with r colors then, for some function f, the vertex set of K can be covered by at most $f(r)$ vertex disjoint monochromatic paths.

In this paper the conjecture is proved in a stronger form. Our main result is

THEOREM 1. *If the edges of a finite complete graph K are colored with r colors then the vertex set of K can be covered by at most $cr^2 \log r$ vertex disjoint monochromatic cycles.*

Theorem 1 makes it possible to define, as a function of r, the minimum number of monochromatic cycles (or paths or trees) needed to cover (or partition) the vertex set of any r-colored complete graph. Problems and

* Partly supported by the AKA Research Fund of the Hungarian Academy of Sciences.

† Research (partially) supported by Hungarian National Foundation for Scientific Research, Grant 1812.

0095-8956/91 $3.00

results concerning these numbers are in Section 3. The strongest conjectures say that the cycle partition number is r and the tree partition number is $r-1$. The latter conjecture is proved for $r=3$ (Theorem 2).

2. Proof of Theorem 1

Lemma 1. *Assume that the edges of the complete bipartite graph (A, B) are colored with r colors. If $|B| \leqslant |A|/r^3$ then B can be covered by at most r^2 vertex disjoint monochromatic cycles.*

Proof. Clearly we can partition B into $B_1, B_2, ..., B_r$ so that for any i, $1 \leqslant i \leqslant r$, $x \in B_i$ is adjacent to at least $|A|/r$ vertices of A in color i. For $x \in B$ let $N_i(x)$ denote the set of vertices in A adjacent to x in color i. Define the graph G_i on vertex set B_i for $i = 1, 2, ..., r$ as follows. For $x, y \in B_i$, xy is an edge of G_i if and only if $|N_i(x) \cap N_i(y)| \geqslant |A|/r^3$.

Claim. The maximum number of pairwise non-adjacent vertices in G_i is at most r for $1 \leqslant i \leqslant r$.

Assume that $x_1, x_2, ..., x_r, x_{r+1} \in B_i$ are pairwise non-adjacent. By the definition of B_i, $|N_i(x_j)| \geqslant |A|/r$ for any j, $1 \leqslant j \leqslant r+1$. Therefore

$$|A| \geqslant \left| \bigcup_{j=1}^{r} N_i(x_j) \right| \geqslant (r+1)\,|A|/r - \sum_{1 \leqslant j < k \leqslant r+1} |N_i(x_j) \cap N_i(x_k)|$$
$$\geqslant |A| \left((r+1)/r - \binom{r+1}{2} \Big/ r^3 \right) > |A|.$$

This contradiction proves the claim. By a theorem of Pósa [14] G_i can be partitioned into at most r vertex disjoint cycles (edges and vertices). Using the definition of G_i and the fact that $|N_i(x) \cap N_i(y)| \geqslant |A|/r^3 \geqslant |B|$, it is easy to find at most r^2 monochromatic vertex disjoint cycles in (A, B) which cover B. ∎

To prepare the proof of Theorem 1, we need a definition. A *triangle cycle* of length k, T_k, is a cycle $a_1, a_2, ..., a_k$ of length k and k further vertices $b_1, b_2, ..., b_k$ such that b_i is adjacent to a_i and to a_{i+1} for $i = 1, 2, ..., k$ $(a_{k+1} = a_1)$. The property of T_k important to us is that T_k has a Hamiltonian cycle after the deletion of any subset of $\{b_1, b_2, ..., b_k\}$. We need the following lemma on the Ramsey number of a triangle cycle.

Lemma 2. *If the edges of K_n are colored with r colors then there exists a monochromatic T_k with $k \geqslant cn/(r(r!)^3)$.*

It is worth noting that $k \geqslant n/f(r)$ follows from a theorem of Chvatal, Rödl, Szemerédi, and Trotter (see in [4]). We give an explicit $f(r)$ in Lemma 2 to provide an explicit $f(r)$ in Theorem 1.

Proof of Lemma 2. It is well known that K_t contains a monochromatic triangle in every r-coloring if $t = 3r!$ (see, for example, [9, p. 127]). This fact implies that in every r-coloring of the edges of K_n there exist at least $\binom{n}{t}/\binom{n-3}{t-3} \geq cn^3/t^3$ monochromatic triangles. We want a subcollection of these triangles such that any two triangles intersect in at most one vertex. We proceed by a greedy algorithm. If m triangles have been selected then at most $3m(n-3)$ further triangles are excluded since one triangle meets at most $3(n-3)$ other triangles in two vertices. The procedure stops if $m + 3m(n-3) > cn^3/t^3$, showing that $m > cn^2/t^3$ (the new c is one-third of the old c). Now we keep only those triangles which are colored with the color used most often, say red, we have at least $cn^2/(rt^3) = s$ red triangles, any two of them meeting in at most one vertex. Remove successively vertices and their incident red triangles if there are less than s/n red triangles incident to the current vertex. It is easy to see that the average red degree does not decrease so we get a non-empty subset X of vertices of K_n such that the red triangles inside X have large minimum degree; i.e., at least $s/n = cn/(rt^3)$ of them are incident to any vertex of X. Let us consider a maximal red triangle path P (defined by analogy with a triangle cycle). If x is an endvertex of P then all the triangles incident to x contain at least one other vertex of P. Taking the nearest vertex from x (on P) for each triangle, we get at least $cn/(rt^3)$ different vertices of P. If y is the one most distant from x (on P) then the red triangle containing x, y and the $x-y$ "subpath" of P determine a red triangle cycle T_k with $k \geq (cn/(r(3r!)^3)$, where the new c is half of the previous c. ∎

THEOREM 1. *If the edges of K_n are colored with r colors then the vertex set of K_n can be covered by at most $cr^2 \log r$ vertex disjoint monochromatic cycles.*

Proof. Assume that K_n is r-colored. By Lemma 2 we can find a monochromatic, say red triangle cycle T_k with $k \geq cn/(r(r!)^3)$. Let X denote the set $\{b_1, b_2, ..., b_k\}$. It is easy to see than an r-colored K_m contains a monochromatic cycle of length at least m/r (by using the most frequent color of K_m and applying the Erdös–Gallai extremal theorem for cycles [5]). Apply repeatedly this fact to the r-colored complete graph induced by $K_n - T_k$. This way choose s vertex disjoint monochromatic cycles in $K_n - T_k$. We wish to choose s such that the set Y of vertices in $K_n - T_k$ uncovered by these s cycles has cardinality at most k/r^3. Since after s steps at most $(n-2k)(1-1/r)^s$ vertices of $K_n - T_k$ are uncovered, we have to choose s to satisfy

$$(n-2k)(1-1/r)^s \leq k/r^3.$$

This inequality is certainly true if

$$(n-2k)(1-1/r)^s \leqslant cn/(r^4(r!)^3)$$

which can be ensured by

$$(1-1/r)^s \leqslant c/(r^4(r!)^3).$$

Elementary calculation shows that $s = \lfloor cr^2 \log r \rfloor$ is a suitable choice for s with some constant c.

Now apply Lemma 1 for the r-colored complete bipartite graph (X, Y). We get a covering of Y by the vertices of at most r^2 vertex disjoint monochromatic cycles. The removal of the vertices of these cycles from $T_k \cup Y$ leaves a red cycle by the definition of X. Thus the vertex set of K_n is partitioned into at most $cr^2 \log r + r^2 + 1$ vertex disjoint monochromatic cycles.

3. Related Results and Open Problems

Define the *cycle partition number* of r-colored complete graphs as the minimum k such that the vertices of any r-colored complete graph can be partitioned into at most k monochromatic cycles. Theorem 1 implies that the cycle partition number depends only on r (and is less than $cr^2 \log r$). Cycle cover, path partition, path cover, tree partition, tree cover numbers can be defined similarly.

The following example shows that the path cover number is at least r. Consider pairwise disjoint sets $A_1, A_2, ..., A_r$ and, for $x \in A_i$, $y \in A_j$, $i \leqslant j$, color the edge xy with color i. If the sequence $|A_i|$ grows fast enough then the vertex set of this r-colored complete graph cannot be covered by less than r monochromatic paths. Perhaps this example is best possible and Theorem 1 can be sharpened as

Conjecture 1. The cycle partition number is r.

This conjecture for $r=2$ is due to J. Lehel. Some special cases for $r=2$ have been solved by Ayel [1]. The cycle cover and path partition numbers are both 2 for $r=2$ [8, 11]. The path partition number of r-colored countable complete graphs is r if paths are understood to be finite or one-way infinite. This is a result of Rado [15].

The rest of this section is devoted to tree cover and tree partition numbers. It is obvious that the tree cover number is at most r since the monochromatic stars at any vertex give a good covering. The following example shows that the tree cover number is at least $r-1$. Consider a complete graph with vertex set identified with the points of an affine plane of order $r-1$. Color the edge pq with color i $(1 \leqslant i \leqslant r)$ if the line through p and q is in the ith parallel class. This example shows that the following conjecture, if true, is best possible.

Conjecture 2. The tree partition number is $r-1$.

The case $r=2$ in Conjecture 2 is equivalent with the fact that for any graph G, either G or its complement is connected, an old remark of Erdös and Rado. The case $r=3$ is settled by

THEOREM 2. *For $r=3$, the tree partition number is* 2.

Proof. Assume that red, blue, and white are the colors of the edges of a complete graph K. A monochromatic connected component is called maximal if its vertex set is not properly contained in any other monochromatic connected component. Let W be a maximal component, assume that it is white. Set $U=V(K)-W$. Consider the complete bipartite graph $B=(W, U)$. The edges of B are red or blue. Let R be a monochromatic, say red component of B. Since W is maximal, $A=R \cap W$ is a proper non-empty subset of W. If $U \cap R$ is a proper subset of U then $(A, U-(U \cap R))$ and $(W-A, U \cap R)$ are two complete blue bipartite graphs with spanning trees satisfying the requirements of the theorem. We may assume therefore that $U \cap R = U$. If for all $x \in A$ there exists $y \in U$ such that xy is blue then the graph of the blue edges spans a connected graph on $V(K)$ and the theorem holds. We may assume therefore that

$$C=\{x \in A\colon xy \text{ is red for all } y \in U\}$$

is a non-empty set. Since W is connected in white, we can write $W=\{x_1, x_2, ..., x_m\}$ with the property that $W-\{x_1, ..., x_i\}$ is connected in white for all i, $i=1, 2, ..., m-1$. Let t be the smallest number with the property that $x_t \in C \cup (W-A)$. If $x_t \in C$ then $\{x_1, ..., x_t\} \cup U$ is connected in red, if $x_t \in W-A$ then $\{x_1, ..., x_t\} \cup U$ is connected in blue. Since $W-\{x_1, ..., x_t\}$ is connected in white, the theorem is proved. ∎

A weaker form of Conjecture 2 is that the tree cover number is $r-1$. This is equivalent to the following conjecture of Lovász and Ryser. An r-partite intersecting hypergraph has a transversal (blocking set) of at most $r-1$ elements. (This is proved by Tuza for $r \leqslant 5$ in [16].) Conjecture 2 implies that an r-colored complete graph K contains a monochromatic tree of at least $|V(K)|/(r-1)$ vertices. This consequence is known to be true [2, 6, 10]. Related problems are also in [3, 7].

The tree partition number seems to be more under control for infinite graphs. A. Hajnal proved [13] that the tree partition number is at most r for infinite r-colored complete graphs and [13] contains further results.

Zs. Nagy and Z. Szentmiklossy proved Theorem 2 for infinite graphs (personal communication).

Partly as a tool to handle the problems formulated above, partly as a problem of its own, it seems interesting to study the case when complete graphs are replaced by complete bipartite graphs. Using Lemma 1, it is possible to show that the cycle cover and tree partition numbers of (an r-colored) $K_{n,n}$ are both at most cr^2. However, we could not prove that the cycle partition number of (an r-colored) $K_{n,n}$ depends only on r.

ACKNOWLEDGMENTS

The authors are grateful to referees whose remarks led to reorganizing the original version of this paper.

REFERENCES

1. J. AYEL, "Sur l'existence de deux cycles supplémentaires unicolorés disjoints et de couleurs différentes dans un graphe complet bicolore," Thesis, University of Grenoble, 1979.
2. A. BIALOSTOCKI AND P. DIERKER, "Monochromatic Connected Subgraphs in a Multicoloring of the Complete Graph," Manuscript.
3. J. BIERBRAUER AND A. GYÁRFÁS, On (n, k)-colorings of complete graphs, *Congr. Numer.* **58** (1987), 123–139.
4. V. CHVATAL, V. RÖDL, E. SZEMERÉDI, AND W. T. TROTTER, The Ramsey number of a graph with bounded maximum degree, *J. Combin. Theory Ser. B* **34** (1983), 239–243.
5. P. ERDŐS AND T. GALLAI, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356.
6. Z. FÜREDI, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981), 155–162.
7. Z. FÜREDI, Covering the complete graph by partitions, submitted for publication.
8. L. GERENCSÉR AND A. GYÁRFÁS, On Ramsey type Problems, *Ann. Univ. Scio. Budapest Eötvös Sect. Math.* **10** (1967), 167–170.
9. R. L. GRAHAM, B. L. ROTHSCHILD, AND J. H. SPENCER, "Ramsey Theory," Wiley, New York, 1980.
10. A. GYÁRFÁS, Partition covers and blocking sets in hypergraphs, *in* "MTA SZTAKI Studies," Vol. 71, 1977 (in Hungarian) (MR58, 5392).
11. A. GYÁRFÁS, Vertex coverings by monochromatic paths and cycles, *J. Graph Theory* **7** (1983), 131–135.
12. A. GYÁRFÁS, Covering complete graphs by monochromatic paths, *in* "Irregularities of Partitions," Algorithms and Combinatorics, Vol. 8, Springer-Verlag, 1989, pp. 89–91.
13. A. HAJNAL, P. KOMJÁTH, L. SOUKUP, AND I. SZALKAI, Decompositions of edge colored infinite complete graphs, *Colloq. Math. Soc. János Bolyai* **52** (1987), 277–280.
14. L. LOVÁSZ, "Combinatorial Problems and Exercises," p. 56, North-Holland, Amsterdam, 1979.
15. R. RADO, Monochromatic paths in graphs, *Ann. Discrete Math.* **3**, 191–194.
16. ZS. TUZA, "On Special Cases of Rysers Conjecture," Manuscript.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium